

**THE WEYL-TYPE ASYMPTOTIC FORMULA FOR BIHARMONIC
STEKLOV EIGENVALUES WITH NEUMANN BOUNDARY
CONDITION IN RIEMANNIAN MANIFOLDS**

GENQIAN LIU

Department of Mathematics, Beijing Institute of Technology, Beijing, the People's
Republic of China. E-mail address: liugqz@bit.edu.cn

ABSTRACT. In this paper, by a new method we establish the Weyl-type asymptotic formula for the counting function of biharmonic Steklov eigenvalues with Neumann boundary condition in a bounded domain of an n -dimensional Riemannian manifold.

1. INTRODUCTION

Let (\mathcal{M}, g) be an oriented Riemannian manifold of dimension n with a positive definite metric tensor g , and let $D \subset \mathcal{M}$ be a bounded domain with C^2 -smooth boundary ∂D . Assume ϱ is a non-negative bounded function defined on ∂D . Consider the following biharmonic Steklov eigenvalue problem with Neumann boundary condition:

$$(1.1) \quad \begin{cases} \Delta_g^2 u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D, \\ \frac{\partial(\Delta_g u)}{\partial \nu} - \lambda^3 \varrho^3 u = 0 & \text{on } \partial D, \end{cases}$$

where ν denotes the inward unit normal vector to ∂D , and Δ_g is the Laplace-Beltrami operator given in local coordinates by

$$\Delta_g = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x_j} \right).$$

Here $|g| := \det(g_{ij})$ is the determinant of the metric tensor, and g^{ij} are the components of the inverse of the metric tensor g .

The problem (1.1) was first discussed in 1968 by J. R. Kuttler and V. G. Sigillito (see [13]) since it describes the deformation u of the linear elastic supported plate D under the action of the transversal exterior force $f(x) = 0$, $x \in D$ (for example, when the weight of the body D is the only body force) with Neumann boundary condition $\frac{\partial u}{\partial \nu}|_D = 0$ (see, [34] or p. 32 of [33]).

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It is well-known that the problem (1.1) has nontrivial solutions u only for a discrete set of $\lambda^3 = \lambda_k^3$, which are called biharmonic Steklov eigenvalues with Neumann boundary condition. Let us enumerate the eigenvalues in increasing order:

$$0 = \lambda_0^3 < \lambda_1^3 \leq \lambda_2^3 \leq \dots \leq \lambda_k^3 \leq \dots,$$

where each eigenvalue is counted as many times as its multiplicity. The corresponding eigenfunctions $u_{01}, u_{02}, u_1, u_2, \dots, u_k, \dots$ form a complete orthonormal basis in $L^2_\varrho(\partial D)$ (see, Proposition 3.5), where $u_{01}(x) = 1$, $u_{02}(x) = \int_D F(x, y) dR_y$ on $x \in \partial D$, and $F(x, y)$ is Green's function in D with Neumann boundary condition. It is clear that λ_k^3 can be characterized variationally as

$$\begin{aligned} \lambda_0 = 0, \quad \lambda_1^3 &= \frac{\int_D |\Delta_g u_1|^2 dR}{\int_{\partial D} \varrho^3 u_1^2 ds} = \inf_{\substack{v \in Lip(\bar{D}) \cap H^2(D) \\ \int_{\partial D} \varrho^3 v ds = 0, \int_{\partial D} \varrho^3 u_{02} v ds = 0}} \frac{\int_D |\Delta_g v|^2 dR}{\int_{\partial D} \varrho^3 v^2 ds}, \\ \lambda_k^3 &= \frac{\int_D |\Delta_g u_k|^2 dR}{\int_{\partial D} \varrho^3 u_k^2 ds} = \max_{\substack{\mathcal{F} \subset \{v \mid v \in Lip(\bar{D}) \cap H^2(D), \\ \int_{\partial D} \varrho^3 v ds = 0, \int_{\partial D} \varrho^3 u_{02} v ds = 0\}, \text{codim}(\mathcal{F}) = k+1}} \inf_{v \in \mathcal{F}} \frac{\int_D |\Delta_g v|^2 dR}{\int_{\partial D} \varrho^3 v^2 ds}, \quad k = 2, 3, \dots \end{aligned}$$

where $H^m(D)$ is the Sobolev space, $Lip(\bar{D})$ is the set of Lipschitz functions on \bar{D} , and where dR and ds are the Riemannian elements of volume and area on D and ∂D , respectively.

The problem (1.1) is also important in biharmonic analysis because the set of the eigenvalues for the biharmonic Steklov problem is the same as the set of eigenvalues of the well-known “Dirichlet to normal derivative of Laplacian” map for biharmonic equation (This map associates each function u defined on the boundary ∂D to the normal derivative $\frac{\partial(\Delta_g u)}{\partial \nu}$ of $\Delta_g u$, where the biharmonic function u in D is uniquely determined by $u|_{\partial D}$ and $(\partial u / \partial \nu)|_{\partial D} = 0$).

In the general case the eigenvalues λ_k^3 can not be evaluated explicitly. In particular, for large k it is difficult to calculate them numerically. In view of the important applications, one is interested in finding the asymptotic formula for λ_k^3 as $k \rightarrow \infty$. Let us introduce the counting function $A(\tau)$ defined as the number of eigenvalues λ_k^3 less than or equal to a given τ^3 . Then our asymptotic problem is reformulated as the study of the asymptotic behavior of $A(\tau)$ as $\tau \rightarrow +\infty$.

In order to better understand our problem (1.1) and its asymptotic behavior, let us mention the Steklov eigenvalue problem for harmonic equation

$$(1.2) \quad \begin{cases} \Delta_g v = 0 & \text{in } D, \\ \frac{\partial v}{\partial \nu} + \iota \varrho v = 0 & \text{on } \partial D, \end{cases}$$

where ι is a real number. This problem was introduced by M. W. Steklov for bounded domains in the plane in [28]. His motivation came from physics. The function v represents the steady state temperature on D such that the flux on the boundary is proportional to the temperature. For the harmonic Steklov eigenvalue problem (1.2), in a special case in two dimensions, Å. Pleijel [23] outlined an investigation of the asymptotic behavior of both eigenvalues ι_k and the eigenfunctions v_k . In 1955, L. Sandgren [26] established the asymptotic formula of the counting function $B(\tau) = \#\{\iota_k \mid \iota_k \leq \tau\}$:

$$(1.3) \quad B(\tau) \sim \frac{\omega_{n-1} \tau^{n-1}}{(2\pi)^{n-1}} \int_{\partial D} \varrho^{n-1} ds \quad \text{as } \tau \rightarrow +\infty,$$

i.e.,

$$\lim_{\tau \rightarrow +\infty} \frac{B(\tau)}{\tau^{n-1}} = \frac{\omega_{n-1}}{(2\pi)^{n-1}} \int_{\partial D} \varrho^{n-1} ds,$$

where ω_{n-1} is the volume of the $(n-1)$ -dimensional unit ball, ds is the Riemannian element of area on ∂D . This asymptotic behavior is motivated by the similar one for the eigenvalues of the Dirichlet-Laplacian. The classical result for the Dirichlet-Laplacian on smooth domain D is Weyl's formula (see [35], [36] or [5]):

$$(1.4) \quad N_D(\tau, D) \sim \frac{\omega_n}{(2\pi)^n} (\text{vol}(D)) \tau^{n/2} \quad \text{as } \tau \rightarrow +\infty,$$

where $N(\tau, D) = \#\{\mu_k \leq \tau\}$ and μ_k is the k -th Dirichlet eigenvalue for D .

The study of asymptotic behavior for the biharmonic Steklov eigenvalues with Neumann boundary condition is much more difficult than that for the harmonic Steklov eigenvalues. It has been a tempting and challenging problem in the past 40 years. The main stumbling block that lies in its way is the estimates for the different kinds of Steklov eigenvalues corresponding to the different kinds of boundary conditions. For the simpler biharmonic Steklov eigenvalue problem with Dirichlet boundary condition, the author established the leading term asymptotic formula of the eigenvalues (see, [16]).

In this paper, for the biharmonic Steklov eigenvalues with Neumann boundary condition, by a new method we establish the Weyl-type asymptotic formula of the counting function. The main results are the following:

Theorem 1.1. *Let (\mathcal{M}, g) be an n -dimensional oriented Riemannian manifold, and let $D \subset M$ be a bounded domain with C^1 -smooth boundary ∂D . Then*

$$(1.5) \quad A(\tau) = \frac{\omega_{n-1} \tau^{n-1}}{(\sqrt[3]{16}\pi)^{n-1}} \int_{\partial D} \varrho^{n-1} ds + o(\tau^{n-1}) \quad \text{as } \tau \rightarrow +\infty,$$

where $A(\tau)$ is defined as before.

Corollary 1.2. *Under hypothesis Theorem 1.1, if $\varrho \equiv 1$ on ∂D for problem (1.1), then*

$$(1.6) \quad \lambda_k \sim \sqrt[3]{16}\pi \left(\frac{k+2}{\omega_{n-1}(\text{vol}(\partial D))} \right)^{1/(n-1)} \quad \text{as } k \rightarrow +\infty.$$

However, when the boundary of a bounded domain is smooth, we have the following Weyl-type asymptotic formula with a better remainder estimate:

Theorem 1.3. *Let (\mathcal{M}, g) be a smooth, n -dimensional oriented Riemannian manifold, and let $D \subset M$ be a bounded domain with smooth boundary ∂D . Then*

$$(1.7) \quad A(\tau) \sim \frac{\omega_{n-1} \tau^{n-1}}{(\sqrt[3]{16}\pi)^{n-1}} \int_{\partial D} \varrho^{n-1} ds + O(\tau^{n-2}) \quad \text{as } \tau \rightarrow +\infty,$$

where $A(\tau)$ is defined as before.

The proofs of our main results use four key techniques: The first technique is the compact trace lemmas for the domain which is the union of a finite number of Lipschitz images of cubes. The second technique is to give the explicit formula for the different kinds of biharmonic Steklov eigenvalues and eigenfunctions in a cube of \mathbb{R}^n (by the method of separation variables we seek the product form of eigenfunctions, one of factors

is the Dirichlet eigenfunction or Neumann eigenfunction, see Section 4). Then we can use the well-known variational methods, which H. Weyl [37] and R. Courant and D. Hilbert [5] have employed in the case of the membrane to give the asymptotic formulas for the two kinds of the Steklov eigenvalues in the cube. The third technique is put the biharmonic Steklov problem into an abstract Hilbert space theory. That is, we first make a division of \bar{D} into subdomains. From this division we construct two Hilbert spaces \mathcal{K}^0 and \mathcal{K}^d and isometric mappings of \mathcal{K}^0 into \mathcal{K} and \mathcal{K} into \mathcal{K}^d . Those of subdomains situated at the boundary Γ_ϱ we can map on cylinders of type treated in Section 4. In a sufficiently fine division of these, the variant of the g^{ik} and ϱ will be small and they can be replaced by constants. Then, we can estimate the asymptotic behavior of the eigenvalues. By means of the results of Section 5, we get the asymptotic formula with leading asymptotic for $A(\tau)$. Finally, applying Theorem 1.1 and a standard technique based on the asymptotic behavior of spectral function of pseudodifferential operator (see, p. 162 of [27], [11] or [31]), we obtain the desired result of Theorem 1.3 (with a better remainder estimate).

2. COMPACT TRACE LEMMAS

An n -dimensional cube in \mathbb{R}^n is the set $\{x \in \mathbb{R}^n | 0 \leq x_k \leq a, k = 1, \dots, n\}$. A set $D \subset \mathbb{R}^n$ is said to be a Lipschitz image of a set $\Omega \subset \mathbb{R}^n$ (see [26]) if there is a one-to-one map from Ω to D defined by

$$(2.1) \quad x = \Psi(x'), \quad x' \in \Omega$$

satisfying a Lipschitz condition

$$(2.2) \quad c^{-1}|x' - y'| \leq |\Psi(x') - \Psi(y')| \leq c|x' - y'|$$

for some constant c and all x' and y' in Ω . ($|x| = (x_1^2 + \dots + x_n^2)^{1/2}$, $|x'| = (x_1'^2 + \dots + x_n'^2)^{1/2}$).

A set $D \subset \mathbb{R}^n$ is said to be star-shaped with respect to a point x^0 if $x \in D$ implies that the closed segment $\{(1-t)x^0 + tx | 0 \leq t \leq 1\}$ is contained in D . Now assume that D is a bounded domain in \mathbb{R}^n and that the closed domain \bar{D} is star-shaped with regard to all points in an open neighborhood of a point $x^0 \in D$. We can assume $x^0 = (0, \dots, 0)$. In this section, $\|x\|$ denotes an arbitrary norm in \mathbb{R}^n with the usual properties of a norm, that is

$$a) \quad \|x\| = 0 \Leftrightarrow x = 0, \quad b) \quad \|x + y\| \leq \|x\| + \|y\| \quad \text{and} \quad c) \quad \|ct\| = |t|\|x\|,$$

where t is a real number. Then evidently (see [26]) there is a $\delta > 0$ such that \bar{D} is star-shaped with respect to all points in $B_\delta = \{x | \|x\| < \delta\}$. Since B_δ is open, it is clearly that $x \in B_\delta$ and $y \in \bar{D}$ implies that all the inner points of the segment $\{(1-t)x + ty | 0 \leq t \leq 1\}$ belong to D .

Lemma 2.1 (Sandgren, p 21 of [26]). *If a bounded domain $\bar{D} \subset \mathbb{R}^n$ is star-shaped with respect to all points in the open cube $\sum_\delta = \{x \in \mathbb{R}^n | \max_{1 \leq i \leq n} |x_i| < \delta\}$, then \bar{D} is a Lipschitz image of the cube \bar{D}' (with the side-length $2a$) given by a transformation*

$$(2.3) \quad x = \Psi(x'), \quad x \in \bar{D} \quad \text{and} \quad x' \in \bar{D}'$$

satisfying the Lipschitz condition

$$(2.4) \quad c^{-1}\|x' - y'\| \leq \|\Psi(x') - \Psi(y')\| \leq c\|x' - y'\|,$$

where $c = \max(3a/\delta, 3b^2/\delta a)$ and $b = \max_{x \in \bar{D}} \|x\|$.

Let f be a real-valued function defined in an open set D in \mathbb{R}^n ($n \geq 1$). For $y \in D$ we call f *real analytic at y* if there exist $a_\beta \in \mathbb{R}^1$ and a neighborhood U of y (all depending on y) such that

$$f(x) = \sum_{\beta} a_\beta (x - y)^\beta$$

for all x in U . We say f is *real analytic in D* , if f is real analytic at each $y \in D$.

From here up to Section 5, let M be an n -dimensional Riemannian manifold with real analytic metric tensor g . We say that \bar{D} is a Lipschitz image of a cube if it is contained in some coordinate neighborhood U and its image \bar{D}_1 in \mathbb{R}^n given by the coordinates of U is a Lipschitz image (see, previous definition) of a closed cube in \mathbb{R}^n .

A subset Γ of (M, g) is said to be an $(n - 1)$ -dimensional smooth surface if Γ is nonempty and if for every point x in Γ , there is a smooth diffeomorphism of the open unit ball $B(0, 1)$ in \mathbb{R}^n onto an open neighborhood U of x such that $B(0, 1) \cap \{x \in \mathbb{R}^n \mid x_n = 0\}$ maps onto $U \cap \Gamma$.

Let D together with its boundary be transformed pointwise into the domain D' together with its boundary by equations of the form

$$(2.5) \quad x'_i = x_i + f_i(x_1, \dots, x_n), \quad i = 1, 2, \dots, n.$$

where the functions f_i and their first order derivatives are Lipschitz continuous throughout the domain, and they are less in absolute value than a small positive number ϵ . Then we say that the domain D is approximated by the domain D' with the degree of accuracy ϵ .

It is well-known (see, for example, p. 133 of [10] or p. 24 of [26]) that every element u in $Lip(\bar{D})$ has partial derivatives $\partial u / \partial x_k$, $k = 1, \dots, n$, which are defined a.e. in D and belong to $L^\infty(D)$. In particular, $Lip(\bar{D}) \subset H^1(D)$.

A subset \mathfrak{F} of $L^2(\partial D)$ is said to be precompact if any infinite sequence $\{u_k\}$ of elements of \mathfrak{F} contains a Cauchy subsequence $\{u_{k'}\}$, i.e., one for which

$$\int_{\partial D} (u_{k'} - u_{l'})^2 ds \rightarrow 0, \quad \text{when } k', l' \rightarrow \infty.$$

Lemma 2.2. *Let $\bar{D} \subset (M, g)$ be a Lipschitz image of a cube and let ϱ be a non-negative function in $L^\infty(\partial D)$ such that $\int_{\partial D} \varrho^3 ds > 0$. Assume \mathfrak{M} is a set of functions u in $\tilde{N}(D) = \{u \mid u \in Lip(\bar{D}) \cap H^2(D), \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D\}$ for which*

$$(2.6) \quad \int_D |\Delta u|^2 dR + \left(\int_{\partial D} \varrho^3 u ds \right)^2$$

is uniformly bounded. Then the set $\{u|_{\partial D} : u \in \mathfrak{M}\}$ is precompact in $L^2(\partial D)$.

Proof. It follows from Theorem 2.3 of [26] that there exists a constant $C_1 > 0$, only depending on D and ϱ , such that for all $u \in Lip(\bar{D})$,

$$(2.7) \quad \int_D u^2 dR \leq C_1 \left[\int_D |\nabla_g u|^2 dR + \left(\int_{\partial D} \varrho^3 u ds \right)^2 \right],$$

where

$$\int_D |\nabla_g u|^2 dR = \int_{\phi(D)} g^{ik}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} \sqrt{|g|} dx,$$

and $\phi(D)$ is the coordinate image of D . Put

$$(2.8) \quad \Lambda_1(D) = \inf_{\substack{u \in \tilde{N}(D) \\ \int_D |\nabla_g u|^2 dR + (\int_{\partial D} \varrho^3 u ds)^2 = 1}} \frac{\int_D |\Delta_g u|^2 dR + (\int_{\partial D} \varrho^3 u ds)^2}{\int_D |\nabla_g u|^2 dR + (\int_{\partial D} \varrho^3 u ds)^2}.$$

In order to prove the existence of a minimizer to (2.8), consider a minimizing sequence u_m in the set $\tilde{N}(D)$, i.e.,

$$\int_D |\Delta_g u_m|^2 dR + (\int_{\partial D} \varrho^3 u_m ds)^2 \rightarrow \Lambda_1(D) \quad \text{as } m \rightarrow +\infty$$

with $\int_D |\nabla_g u_m|^2 dx + (\int_{\partial D} \varrho^3 u_m ds)^2 = 1$. Thus, there is a constant $C_2 > 0$ such that

$$(2.9) \quad \|\Delta_g u_m\|_{L^2(D)} \leq C_2, \quad \int_D |\nabla_g u_m|^2 dx \leq C_2, \quad (\int_{\partial D} \varrho^3 u_m ds)^2 \leq C_2$$

for all $m \geq 1$. It follows from the *a priori* estimate for elliptic equations (see, for example, Proposition 7.2 of p.345 in [30]) that there exists a constant $C_3 > 0$ depending only on n, D such that

$$(2.10) \quad \|u_m\|_{H^2(D)} \leq C_4 (\|\Delta_g u_m\|_{L^2(D)} + \|u_m\|_{H^1(D)}).$$

From this, (2.7) and (2.9), we have that

$$\|u_m\|_{H^2(D)} \leq C_4 \quad \text{for all } m.$$

By the Banach-Alaoglu theorem we can then extract a subsequence, which we still call $\{u_m\}$, converging weakly in $H^2(D)$ to a limit u , and strongly converging to u in $L^2(D)$. Since the functional $\int_D |\Delta_g u|^2 dR$ is lower semicontinuous in the weak $H^2(D)$ topology, we have

$$\int_D |\Delta_g u|^2 dR \leq \liminf_{m \rightarrow \infty} \int_D |\Delta_g u_m|^2 dR,$$

Since $u_m \rightarrow u$ weakly in $H^2(D)$, we get that $u_m \rightarrow u$ strongly in $H^r(D_l)$ for any $0 < r < 2$. Note that $\frac{\partial u_m}{\partial \nu}|_{\partial D} = 0$. It follows that $\frac{\partial u}{\partial \nu}|_{\partial D} = 0$. Therefore $u \in \mathfrak{M}$ is a minimizer.

We claim that $\Lambda_1(D) > 0$. Suppose by contradiction that $\Lambda_1(D) = 0$. Then

$$(2.11) \quad \begin{cases} \Delta_g u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D \end{cases}$$

and

$$(2.12) \quad \int_{\partial D} \varrho^3 u ds = 0.$$

The boundary value problem (2.11) implies that $u \equiv \text{constant}$ in D . By $\int_{\partial D} \varrho^3 ds > 0$ and (2.12), we then get $u = 0$ in D . This contradicts the fact that $\int_D |\nabla_g u|^2 dR + (\int_{\partial D} \varrho^3 u ds)^2 = 1$, and the claim is proved.

By (2.8), we obtain that for every $u \in \tilde{N}(D)$,

$$(2.13) \quad \int_D |\nabla_g u|^2 dR + (\int_{\partial D} \varrho^3 u ds)^2 \leq \frac{1}{\Lambda_1(D)} \left(\int_D |\Delta_g u|^2 dR + (\int_{\partial D} \varrho^3 u ds)^2 \right).$$

It follows from (2.7) and (2.13) that \mathfrak{M} is a bounded set in $H^1(D)$. Since $\bar{D} \subset (M, g)$ is a Lipschitz image of a cube, it follows from [2] (see also, Chs V, VI of [6]) that the set $\{u|_{\partial D} : u \in \mathfrak{M}\}$ is precompact in $L^2(\partial D)$. \square

Lemma 2.3. *Let (\mathcal{M}, g) be a real analytic Riemannian manifold, and let $\bar{D} \subset (M, g)$ be a Lipschitz image of a cube. Assume that Γ_1 is a portion of ∂D and that Γ_0 is an $(n-1)$ -dimensional $C^{2,\varepsilon}$ -smooth surface in ∂D satisfying $\Gamma_0 \subset \subset \partial D - \Gamma_1$. Suppose that \tilde{q} is a non-negative function defined on ∂D and assume that \mathfrak{E} (respectively, \mathfrak{S}) is a set of functions u in $K^d(D) = \{u | u \in \text{Lip}(\bar{D}) \cap H^2(D), \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1, u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0\}$ (respectively, $K(D) = \{u | u \in \text{Lip}(\bar{D}) \cap H^2(D), \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1 \cup \Gamma_0, \text{ and } u = 0 \text{ on } \partial D - \Gamma_1\}$) for which*

$$(2.14) \quad \int_D |\Delta_g u|^2 dR + \left(\int_{\partial D} \tilde{q}^3 u ds \right)^2$$

is uniformly bounded. Then the set $\{u|_{\partial D} : u \in \mathfrak{E}\}$ (respectively, $\{u|_{\partial D} : u \in \mathfrak{S}\}$) is precompact in $L^2(\partial D)$.

Proof. We only prove the case in $K^d(D)$ because the method is similar for the case in $K(D)$. It follows from Theorem 2.3 of [26] that there exists a constant $C > 0$, depending only on D and \tilde{q} , such that for every $u \in \text{Lip}(\bar{D})$,

$$(2.15) \quad \int_D u^2 dR \leq C \left[\int_D |\nabla_g u|^2 dR + \left(\int_{\partial D} \tilde{q}^3 u ds \right)^2 \right].$$

Let

$$(2.16) \quad \Lambda_{\tilde{q}}(D) = \inf_{\substack{v \in K^d(D) \\ \int_D |\nabla_g v|^2 dR + (\int_{\partial D} \tilde{q}^3 v ds)^2 = 1}} \frac{\int_D |\Delta_g v|^2 dR + (\int_{\partial D} \tilde{q}^3 v ds)^2}{\int_D |\nabla_g v|^2 dR + (\int_{\partial D} \tilde{q}^3 v ds)^2}.$$

In order to prove the existence of a minimizer to (2.16), consider a minimizing sequence v_m in $K^d(D)$, i.e.,

$$\int_D |\Delta_g v_m|^2 dR + \left(\int_{\partial D} \tilde{q}^3 v_m ds \right)^2 \rightarrow \Lambda_{\tilde{q}}(D) \quad \text{as } m \rightarrow +\infty$$

with $\int_D |\nabla_g v_m|^2 dR + (\int_{\partial D} \tilde{q}^3 v_m ds)^2 = 1$. Thus, there is a constant $\tilde{C} > 0$ such that

$$(2.17) \quad \begin{aligned} \|\Delta_g v_m\|_{L^2(D)}^2 + \left(\int_{\partial D} \tilde{q}^3 v_m ds \right)^2 &\leq \tilde{C}, \\ \|\nabla_g v_m\|_{L^2(D)}^2 + \left(\int_{\partial D} \tilde{q}^3 v_m ds \right)^2 &\leq \tilde{C} \quad \text{for all } m. \end{aligned}$$

Let $\{D_l\}$ be a sequence of Lipschitz domains such that $D_1 \subset D_2 \subset \cdots \subset D_l \subset \cdots \subset \subset (D \cup \Gamma_1 \cup \Gamma_0)$, $\cup_{l=1}^{\infty} D_l = D$, and $\Gamma_1 \cup \Gamma_0 \subset \partial D_l$ for all l . It follows from the *a priori* estimate for elliptic equations (see, for example, the proof of Proposition 7.2 of [30]) that there exists a constant $C'_l > 0$ depending only on n, D_l, D, Γ_1 and Γ_0 such that

$$(2.18) \quad \|v_m\|_{H^2(D_l)} \leq C'_l (\|\Delta_g v_m\|_{L^2(D)} + \|v_m\|_{H^1(D)}).$$

From this, (2.15) and (2.17), we have that

$$\|v_m\|_{H^2(D_l)} \leq C''_l \quad \text{for all } m,$$

where C_l'' is a constant. For each l , by the Banach-Alaoglu theorem we can extract a subsequence $\{v_{l,m}\}_{m=1}^\infty$ of $\{v_m\}$, which converges weakly in $H^2(D_l)$ to a limit u , and strongly converges to u in $L^2(D_l)$. We may assume that $\{v_{l+1,m}\}$ is a subsequence of $\{v_{l,m}\}$ for every l . Then, the diagonal sequence $\{v_{l,l}\}$ converges weakly in H^2 to u , and strongly converges to u in L^2 , in every compact subset E of D . It is obvious that $\|\nabla_g u\|_{L^2(D)}^2 + \left(\int_{\partial D} \tilde{\varrho}^3 u ds\right)^2 = 1$. Since the functional $\int_{D_l} |\Delta_g u|^2 dR$ is lower semicontinuous in the weak $H^2(D_l)$ topology, we have

$$\int_{D_l} |\Delta_g u|^2 dR \leq \varliminf_{k \rightarrow \infty} \int_{D_l} |\Delta_g v_{k,k}|^2 dR,$$

so that

$$\begin{aligned} \int_D |\Delta_g u|^2 dR &= \lim_{l \rightarrow \infty} \int_{D_l} |\Delta_g u|^2 dR \leq \lim_{l \rightarrow \infty} \left(\varliminf_{k \rightarrow \infty} \int_{D_l} |\Delta_g v_{k,k}|^2 dR \right) \\ &\leq \lim_{l \rightarrow \infty} \left(\varliminf_{k \rightarrow \infty} \int_D |\Delta_g v_{k,k}|^2 dR \right) = \varliminf_{k \rightarrow \infty} \int_D |\Delta_g v_{k,k}|^2 dR. \end{aligned}$$

In addition, for each fixed l , since $v_{k,k} \rightarrow u$ weakly in $H^2(D_l)$, we get that $v_{k,k} \rightarrow u$ strongly in $H^r(D_l)$ for any $0 < r < 2$. Note that $\frac{\partial v_{k,k}}{\partial \nu}|_{\Gamma_1} = 0$ and $v_{k,k}|_{\Gamma_0} = \frac{\partial v_{k,k}}{\partial \nu}|_{\Gamma_0} = 0$. It follows that $\frac{\partial u}{\partial \nu}|_{\Gamma_1} = 0$ and $u|_{\Gamma_0} = \frac{\partial u}{\partial \nu}|_{\Gamma_0} = 0$. Therefore $u \in K^d(D)$ is a minimizer.

We claim that $\Lambda_{\tilde{\varrho}}(D) > 0$. Suppose by contradiction that

$$\Lambda_{\tilde{\varrho}}(D) = \frac{\int_D |\Delta_g u|^2 dR + \left(\int_{\partial D} \tilde{\varrho}^3 u ds\right)^2}{\int_D |\nabla_g u|^2 dR + \left(\int_{\partial D} \tilde{\varrho}^3 u ds\right)^2} = 0.$$

It follows that $\Delta_g u = 0$ in D . Since the coefficients of the Laplacian are real analytic in D , and since Γ_0 is a $C^{2,\varepsilon}$ -smooth surface, we find with the aid of the regularity for elliptic equations (see, Theorem A of [19], [18] or [1]) that u is $C^{2,\varepsilon}$ -smooth up to the partial boundary Γ_0 . Note that $u = \frac{\partial u}{\partial \nu} = 0$ on Γ_0 . Applying Holmgren's uniqueness theorem (see, Corollary 5 of p. 39 in [24]) for the real analytic elliptic equation $\Delta_g u = 0$ in D , we get that $u \equiv 0$ in D . This contradicts the fact $\int_D |\nabla_g u|^2 dR + \left(\int_{\partial D} \tilde{\varrho}^3 u ds\right)^2 = 1$, and the claim is proved. Therefore we have that

$$\begin{aligned} (2.19) \quad & \int_D |\nabla_g u|^2 dR + \left(\int_{\partial D} \tilde{\varrho}^3 u ds\right)^2 \\ & \leq \frac{1}{\Lambda_{\tilde{\varrho}}} \left[\int_D |\Delta_g u|^2 dR + \left(\int_{\partial D} \tilde{\varrho}^3 u ds\right)^2 \right] \quad \text{for } u \in K^d(D). \end{aligned}$$

According to the assumption, there is a constant C'' such that

$$\|\Delta_g u\|_{L^2(D)}^2 + \left(\int_{\partial D} \tilde{\varrho}^3 u ds\right)^2 \leq C'' \quad \text{for all } u \in \mathfrak{E},$$

and hence

$$(2.20) \quad \|\nabla_g u\|_{L^2(D)}^2 + \left(\int_{\partial D} \tilde{\varrho}^3 u ds\right)^2 \leq C'' \quad \text{for all } u \in \mathfrak{E}.$$

Combining this and (2.15), we have

$$\|u\|_{H^1(D)} \leq C, \quad \text{for all } u \in \mathfrak{E},$$

which implies that $\{u|_{\partial D} : u \in \mathfrak{E}\}$ is precompact in $L^2(\partial D)$. \square

Corollary 2.4. *Lemma 2.3 is still true if we exchange (2.14) for*

$$(2.21) \quad \int_D |\Delta_g u|^2 dR.$$

Proof. Let $\tilde{\varrho} \in L^\infty(\partial D)$ be the characteristic functions of Γ_0 (respectively, $\Gamma - \Gamma_1$) in the case of $K^d(D)$ (respectively, $K(D)$). Then $\int_\Gamma \tilde{\varrho}^3 u ds = 0$ for all u in \mathfrak{M} , and hence according to (2.21), the condition (2.14) is satisfied such that Lemma 2.3 can be applied. \square

Let $\bar{D}_1, \bar{D}_2 \subset (M, g)$ be Lipschitz images of cubes with boundaries ∂D_1 and ∂D_2 . We say an open domain D with boundary ∂D and closure $\bar{D} = D \cup \partial D$ is composed of \bar{D}_1 and \bar{D}_2 if

- a) $\partial D_1 \cap \partial D_2$ has positive measure
- b) $\bar{D} = \bar{D}_1 \cup \bar{D}_2$
- c) $\partial D \subset \partial D_1 \cup \partial D_2$
- d) ∂D has positive measure.

The requirement d) excludes for instance the possibility that M is a sphere and $D = M$ the union of two hemispheres \bar{D}_1 and \bar{D}_2 (see, p.27 of [26]).

By a finite number of domains, each of which is a Lipschitz image of a cube, we can obtain more domains according to the above method. Denoted by \mathcal{F} all such domains. Completely similar to the proofs of Lemmas 2.2, 2.3, we find that the compact trace Lemmas 2.2, 2.3 are also true for each domain in class \mathcal{F} .

3. SOME COMPLETELY CONTINUOUS TRANSFORMATIONS AND THEIR EIGENVALUES

Let (\mathcal{M}, g) be an n -dimensional real analytic Riemannian manifold and let $D \subset M$ be a bounded domain with boundary Γ . Suppose that D is of the type defined in Section 2 (i.e., $D \in \mathcal{F}$) so that the compact trace lemmas 2.2, 2.3 are true. Let ϱ be a non-negative bounded function defined on Γ or only on a portion Γ_ϱ of Γ (measure $\Gamma_\varrho = \int_{\Gamma_\varrho} ds > 0$) and assume that $\int_{\Gamma_\varrho} \varrho^3 ds > 0$. In the case $\Gamma_\varrho \neq \Gamma$ we let Γ_0 be a $C^{2,\varepsilon}$ -smooth $(n-1)$ -dimensional surface in $\Gamma - \bar{\Gamma}_\varrho$.

If $\Gamma_\varrho \neq \Gamma$ (measure $\Gamma - \Gamma_\varrho > 0$), we denote

$$K(D) = \{u | u \in Lip(\bar{D}) \cap H^2(D), \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_\varrho \cup \Gamma_0, \text{ and } u = 0 \text{ on } \Gamma - \Gamma_\varrho\},$$

$$K^d(D) = \{u | u \in Lip(\bar{D}) \cap H^2(D), \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_\varrho, \text{ and } u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0\}.$$

If $\Gamma_\varrho = \Gamma$, we denote

$$N(D) = \{u | u \in Lip(\bar{D}) \cap H^2(D), \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma, \int_\Gamma \varrho^3 u u_{01} ds = 0 \text{ and } \int_\Gamma \varrho^3 u u_{02} ds = 0\},$$

where $u_{01} = 1$ and u_{02} are two eigenfunctions corresponding to the Steklov eigenvalue $\lambda = 0$ (see, Section 1). We shall also use the notation

$$\langle u, v \rangle^* = \int_D (\Delta_g u)(\Delta_g v) dR, \quad u, v \in K(D) \text{ or } L^d(D) \text{ or } N(D).$$

The bilinear functional $\langle u, v \rangle^*$ can be used as an inner product in each of the spaces $K(D)$, $K^d(D)$ and $N(D)$. In fact, $\langle u, v \rangle^*$ is a positive, symmetric, bilinear functional. In addition, if $\langle u, u \rangle^* = 0$, then $\Delta_g u = 0$ in D . For $u \in K(D)$ or $N(D)$, by applying Green's formula, we have

$$0 = \int_D u(\Delta_g u) dR = - \int_D |\nabla_g u|^2 dR - \int_{\partial D} u \frac{\partial u}{\partial \nu} ds = - \int_D |\nabla_g u|^2 dR,$$

which implies $u \equiv \text{constant}$ in D . In the case $u \in K(D)$, in view of $u = 0$ on $\Gamma - \Gamma_\varrho$, we get that $u \equiv 0$ in D ; In the case $u \in N(D)$, since $\int_D \varrho^3 u ds = 0$ with $\int_\Gamma \varrho^3 ds > 0$, we obtain $u \equiv 0$ in D ; In the case $u \in K^d(D)$, since Γ_0 is a $C^{2,\varepsilon}$ -smooth surface and $u = \frac{\partial u}{\partial \nu} = 0$ on Γ_0 , we find by $\Delta u = 0$ in D and Holmgren's uniqueness theorem (see, Corollary 5 of p.39 in [24]) that $u \equiv 0$ in D . Closing $K(D)$, $K^d(D)$ and $N(D)$ with respect to the norm $\|u\|^\star = \sqrt{\langle u, u \rangle^*}$, we get the Hilbert spaces $(\mathcal{K}, \|\cdot\|^\star)$, $(\mathcal{K}^d, \|\cdot\|^\star)$ and $(\mathcal{N}, \|\cdot\|^\star)$, respectively.

Next, we consider two linear functionals

$$[u, v] = \int_{\Gamma_\varrho} \varrho^3 uv ds$$

and

$$(3.1) \quad \langle u, v \rangle = \langle u, v \rangle^* + [u, v],$$

where $u, v \in K(D)$ or $u, v \in K^d(D)$ or $u, v \in N(D)$. It is clear that $\langle u, v \rangle$ is an inner product in each of the spaces $K(D)$, $K^d(D)$ and $N(D)$.

Lemma 3.1. *The norm*

$$\|u\|^\star = \sqrt{\langle u, u \rangle^*}$$

and

$$\|u\| = \sqrt{\langle u, u \rangle}$$

are equivalent in $K(D)$, $K^d(D)$ and $N(D)$.

Proof. We have to show that there exist positive constants C_1 and C_2 such that

$$C_1 \|u\|^\star \leq \|u\| \leq C_2 \|u\|^\star \quad \text{for all } u \text{ in } K(D) \text{ or } K^d(D) \text{ or } N(D).$$

Obviously, $\|u\|^\star \leq \|u\|$ for all u in each of three spaces. Let us first consider the case $u \in N(D)$. It suffices to show that $\|u\|$ is bounded when u belongs to the set

$$\mathfrak{M} = \{u | u \in N(D), \|u\|^\star \leq 1\}.$$

It follows from Lemma 2.2 that $\mathfrak{M}_\Gamma := \{u|_\Gamma : u \in \mathfrak{M}\}$ is precompact in $L^2(\Gamma)$. This implies that there exists a constant $C > 0$ such that $\int_\Gamma u^2 ds \leq C$ for all $u \in \mathfrak{M}$. Therefore, $[u, u] = \int_\Gamma \varrho^3 u^2 ds$ is bounded in \mathfrak{M} , and so is $\|u\|^2 = \langle u, u \rangle^* + [u, u]$.

Next, we consider the case for $u \in K(D)$ and still denote

$$\mathfrak{M} = \{u | u \in K(D), \|u\|^\star \leq 1\}.$$

By taking

$$\tilde{\varrho} = \begin{cases} 1 & \text{for } x \in \Gamma - \Gamma_\varrho, \\ 0 & \text{for } x \in \Gamma_\varrho, \end{cases}$$

we have

$$(3.2) \quad \int_D |\Delta u|^2 dR + \left(\int_{\Gamma} \tilde{\varrho}^3 u ds \right)^2 \leq C \quad \text{for every } u \in \mathfrak{M}.$$

It follows from Lemma 2.3 that \mathfrak{M}_{Γ} is precompact in $L^2(\Gamma)$. In particular, $\int_{\Gamma_e} u^2 ds$ is bounded on \mathfrak{M} and hence also $[u, u] = \int_{\Gamma_e} \varrho^3 u^2 ds$ and $\|u\| = \|u\|^{*} + [u, u]$.

Similarly, by taking $\tilde{\varrho}$ to be the characteristic function of Γ_0 and by applying Lemma 2.3, we can prove the corresponding results for the space $K^d(D)$. \square

From Lemma 3.1, it follows that

$$|[u, u]| = \left| \int_{\Gamma_e} \varrho^3 u^2 ds \right| \leq C \langle u, u \rangle^{*} \quad \text{for all } u \text{ in } K(D) \text{ or } K^d(D) \text{ or } N(D).$$

Therefore, $[u, v]$ is a bounded, symmetric, bilinear functional in $(K(D), \langle \cdot, \cdot \rangle^{*})$, $(K^d(D), \langle \cdot, \cdot \rangle^{*})$ and $(N(D), \langle \cdot, \cdot \rangle^{*})$. Since it is densely defined in $(\mathcal{K}, \langle \cdot, \cdot \rangle^{*})$, $(\mathcal{K}^d, \langle \cdot, \cdot \rangle^{*})$ and $(\mathcal{N}, \langle \cdot, \cdot \rangle^{*})$, respectively, it can immediately be extended to $(\mathcal{K}, \langle \cdot, \cdot \rangle^{*})$, $(\mathcal{K}^d, \langle \cdot, \cdot \rangle^{*})$ and $(\mathcal{N}, \langle \cdot, \cdot \rangle^{*})$. We still use $[u, v]$ to express the extended functional. Then there is a bounded linear transformation $G_{\mathcal{K}}^{(*)}$ of $(\mathcal{K}, \langle \cdot, \cdot \rangle^{*})$ into $(\mathcal{K}, \langle \cdot, \cdot \rangle^{*})$ (respectively, $G_{\mathcal{K}^d}^{(*)}$ of $(\mathcal{K}^d, \langle \cdot, \cdot \rangle^{*})$ into $(\mathcal{K}^d, \langle \cdot, \cdot \rangle^{*})$, $G_{\mathcal{N}}^{(*)}$ of $(\mathcal{N}, \langle \cdot, \cdot \rangle^{*})$ into $(\mathcal{N}, \langle \cdot, \cdot \rangle^{*})$) such that

$$(3.3) \quad [u, v] = \langle G_{\mathcal{K}}^{(*)} u, v \rangle^{*} \quad \text{for all } u \text{ and } v \text{ in } \mathcal{K}$$

(respectively,

$$(3.4) \quad [u, v] = \langle G_{\mathcal{K}^d}^{(*)} u, v \rangle^{*} \quad \text{for all } u \text{ and } v \text{ in } \mathcal{K}^d,$$

$$(3.5) \quad [u, v] = \langle G_{\mathcal{N}}^{(*)} u, v \rangle^{*} \quad \text{for all } u \text{ and } v \text{ in } \mathcal{N}.$$

Lemma 3.2. *The transformations $G_{\mathcal{K}}^{(*)}$, $G_{\mathcal{K}^d}^{(*)}$ and $G_{\mathcal{N}}^{(*)}$ are self-adjoint and compact.*

Proof. Since $[u, v]$ is symmetric, we immediately get that the transformation $G_{\mathcal{K}}^{(*)}$, $G_{\mathcal{K}^d}^{(*)}$ and $G_{\mathcal{N}}^{(*)}$ are all self-adjoint. For the compactness, we only discuss the case for the transformation $G_{\mathcal{K}}^{(*)}$. It suffices to show (see, p. 204 of [25]): From every sequence $\{u_m\}$ in $K(D)$ which is bounded

$$(3.6) \quad \|u_m\|^{*} \leq \text{constant}, \quad m = 1, 2, 3, \dots,$$

we can pick out a subsequence $\{u_{m'}\}$ such that

$$(3.7) \quad \langle G_{\mathcal{K}}^{(*)} (u_{m'} - u_{l'}), (u_{m'} - u_{l'}) \rangle^{*} \rightarrow 0 \quad \text{when } m', l' \rightarrow \infty.$$

Applying Lemmas 2.3, 3.1 with the aid of (3.6), we find that the sequence $\{u_m|_{\Gamma_e}\}$ is precompact in $L^2(\Gamma_e)$, so that there is a subsequence $\{u_{m'}\}$ such that

$$\int_{\Gamma_e} (u_{m'} - u_{l'})^2 ds \rightarrow 0 \quad \text{as } m', l' \rightarrow \infty.$$

Therefore

$$[u_{m'} - u_{l'}, u_{m'} - u_{l'}] = \int_{\Gamma_e} \varrho^3 (u_{m'} - u_{l'})^2 ds \rightarrow 0 \quad \text{as } m', l' \rightarrow \infty,$$

which implies (3.7). This proves the compactness of $G_{\mathcal{K}}^{(*)}$. \square

Except for the transformations $G_{\mathcal{K}}^{(*)}$, $G_{\mathcal{K}^d}^{(*)}$ and $G_{\mathcal{N}}^{(*)}$, we need introduce corresponding transformations $G_{\mathcal{K}}$, $G_{\mathcal{K}^d}$ and $G_{\mathcal{N}}$ by the inner product $\langle \cdot, \cdot \rangle$. Since

$$(3.8) \quad 0 \leq [u, v] \leq \langle u, v \rangle \quad \text{for all } u \text{ in } K(D) \text{ or } K^d(D) \text{ or } N(D),$$

there is a bounded linear self-adjoint transformation $G_{\mathcal{K}}$ of $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ (respectively, $G_{\mathcal{K}^d}$ of $(\mathcal{K}^d, \langle \cdot, \cdot \rangle)$, $G_{\mathcal{N}}$ of $(\mathcal{N}, \langle \cdot, \cdot \rangle)$) such that

$$(3.9) \quad [u, v] = \langle G_{\mathcal{K}} u, v \rangle \quad \text{for all } u \text{ and } v \text{ in } \mathcal{K}$$

(respectively,

$$(3.10) \quad [u, v] = \langle G_{\mathcal{K}^d} u, v \rangle \quad \text{for all } u \text{ and } v \text{ in } \mathcal{K}^d,$$

$$(3.11) \quad [u, v] = \langle G_{\mathcal{N}} u, v \rangle \quad \text{for all } u \text{ and } v \text{ in } \mathcal{N}).$$

Lemma 3.3. *The transformations $G_{\mathcal{K}}$, $G_{\mathcal{K}^d}$ and $G_{\mathcal{N}}$ are positive and compact.*

Proof. From $[u, u] \geq 0$ for any $u \in \mathcal{K}$ or \mathcal{K}^d or \mathcal{N} , we immediately know that $G_{\mathcal{K}}$, $G_{\mathcal{K}^d}$ and $G_{\mathcal{N}}$ are positive. The proof of the compactness is completely similar to that of Lemma 3.2. \square

It follows from Lemma 3.3 that $G_{\mathcal{K}}$ (respectively, $G_{\mathcal{K}^d}$, $G_{\mathcal{N}}$) has only non-negative eigenvalues and that the positive eigenvalues form an enumerable sequence $\{\mu_{\mathcal{K}}\}$ (respectively, $\{\mu_{\mathcal{K}^d}\}$, $\{\mu_{\mathcal{N}}\}$) with 0 as the only limit point.

Theorem 3.4. *The transformations $G_{\mathcal{K}}^{(*)}$ and $G_{\mathcal{K}}$ (respectively, $G_{\mathcal{K}^d}^{(*)}$ and $G_{\mathcal{K}^d}$, $G_{\mathcal{N}}^{(*)}$ and $G_{\mathcal{N}}$) have the same eigenfunctions. If $\mu_{\mathcal{K}}^*$ and $\mu_{\mathcal{K}}$ (respectively, $\mu_{\mathcal{K}^d}^*$ and $\mu_{\mathcal{K}^d}$, $\mu_{\mathcal{N}}^*$ and $\mu_{\mathcal{N}}$) are eigenvalues corresponding to the same eigenfunction we have*

$$(3.12) \quad \mu_{\mathcal{K}} = \frac{\mu_{\mathcal{K}}^*}{1 + \mu_{\mathcal{K}}^*}$$

(respectively,

$$(3.13) \quad \mu_{\mathcal{K}^d} = \frac{\mu_{\mathcal{K}^d}^*}{1 + \mu_{\mathcal{K}^d}^*},$$

$$(3.14) \quad \mu_{\mathcal{N}} = \frac{\mu_{\mathcal{N}}^*}{1 + \mu_{\mathcal{N}}^*}.$$

Proof. We only prove the case for the $G_{\mathcal{K}}$ (the arguments are similar for $G_{\mathcal{K}^d}$ and $G_{\mathcal{N}}$). Since $G_{\mathcal{K}}^{(*)}$ is positive, we can easily conclude that the inverse $(1 + G_{\mathcal{K}}^{(*)})^{-1}$ exists and is a bounded self-adjoint transformation. By virtue of (3.3), (3.9) and (3.1), we have

$$(3.15) \quad \begin{aligned} \langle G_{\mathcal{K}}^{(*)} u, v \rangle^* &= [u, v] = \langle G_{\mathcal{K}} u, v \rangle \\ &= \langle G_{\mathcal{K}} u, v \rangle^* + \langle G_{\mathcal{K}}^{(*)} G_{\mathcal{K}} u, v \rangle^* \\ &= \langle G_{\mathcal{K}} u, v \rangle^* + \langle G_{\mathcal{K}} G_{\mathcal{K}}^{(*)} u, v \rangle^*, \quad (u, v \in \mathcal{K}). \end{aligned}$$

It follows that

$$(3.16) \quad G_{\mathcal{K}} = G_{\mathcal{K}}^{(*)} (1 + G_{\mathcal{K}}^{(*)})^{-1},$$

from which the desired result follows immediately. \square

Proposition 3.5. *Let u and v be two eigenfunctions in $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ (respectively, $(\mathcal{K}^d, \langle \cdot, \cdot \rangle)$, $(\mathcal{N}, \langle \cdot, \cdot \rangle)$) of the transformation $G_{\mathcal{K}}$ (respectively, $G_{\mathcal{K}^d}$, $G_{\mathcal{N}}$) at least one of which corresponds to a non-vanishing eigenvalue. Then u and v are orthogonal if and only if the $u|_{\Gamma_e}$ and $v|_{\Gamma_e}$ are orthogonal in $L^2_e(\Gamma_e)$, that is,*

$$(3.17) \quad [u, v] = \int_{\Gamma_e} \varrho^3 uv \, ds = 0.$$

Proof. Without loss of generality, we suppose that u is the eigenfunction corresponding to the eigenvalue $\mu \neq 0$. Then

$$[u, v] = \langle G_{\mathcal{K}} u, v \rangle = \mu \langle u, v \rangle,$$

which implies the desired result. \square

We can now prove

Theorem 3.6. *Let $D \subset (\mathcal{M}, g)$ be a bounded domain with piecewise smooth boundary Γ , and let $D \in \mathcal{F}$. If u is an eigenfunction of the transformations $G_{\mathcal{K}}^{(\star)}$ or $G_{\mathcal{N}}^{(\star)}$ with eigenvalue $\mu^{\star} \neq 0$, then u has derivatives of any order in D and is such that*

$$(3.18) \quad \begin{cases} \Delta_g^2 u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_e \cup \Gamma_0, \quad u = 0 & \text{on } \Gamma - \Gamma_e, \\ \Delta_g u = 0 & \text{on } \Gamma - (\Gamma_e \cup \Gamma_0), \\ \frac{\partial(\Delta_g u)}{\partial \nu} - \lambda^3 \varrho^3 u = 0 & \text{on } \Gamma_e, \quad \text{with } \lambda^3 = \frac{1}{\mu^{\star}}. \end{cases}$$

Proof. Since u is an eigenfunction of $G_{\mathcal{K}}^{(\star)}$ (i.e., $G_{\mathcal{K}}^{(\star)} u = \mu^{\star} u$), we have that $\frac{\partial u}{\partial \nu}|_{\Gamma_e \cup \Gamma_0} = 0$ and $u|_{\Gamma - \Gamma_e} = 0$, and that

$$\int_{\Gamma_e} \varrho^3 uv \, ds = \mu^{\star} \int_D (\Delta_g u)(\Delta_g v) dR \quad \text{for all } v \in \mathcal{K}(D).$$

Applying Green's formula (see, p. 114-120 of [15], [3]) to the right-hand side of the above equation, we obtain that

$$\frac{1}{\mu^{\star}} \int_{\Gamma_e} \varrho^3 uv \, ds = \int_D (\Delta_g^2 u) v \, dR - \int_{\Gamma} (\Delta_g u) \frac{\partial v}{\partial \nu} \, ds + \int_{\Gamma} \frac{\partial(\Delta_g u)}{\partial \nu} v \, ds$$

for all $v \in \mathcal{K}(D)$, where $\frac{\partial(\Delta_g u)}{\partial \nu} \in H^{-\frac{3}{2}}(\Gamma)$ (see [15]). From $\frac{\partial v}{\partial \nu}|_{\Gamma_e \cup \Gamma_0} = 0$ and $v|_{\Gamma - \Gamma_e} = 0$, we get

$$(3.19) \quad \int_D (\Delta_g^2 u) v \, dR - \int_{\Gamma - (\Gamma_e \cup \Gamma_0)} (\Delta_g u) \frac{\partial v}{\partial \nu} \, ds + \int_{\Gamma_e} \left(\frac{\partial(\Delta_g u)}{\partial \nu} - \frac{1}{\mu^{\star}} \varrho^3 u \right) v \, ds = 0$$

for all $v \in \mathcal{K}(D)$. By taking all $v \in C_0^{\infty}(D)$, we have $\Delta_g^2 u = 0$ in D . It follows from the interior regularity of elliptic equations that $u \in C^{\infty}(D)$. Noticing that $v|_{\Gamma_e}$ and $\frac{\partial v}{\partial \nu}|_{\Gamma - (\Gamma_e \cup \Gamma_0)}$ run throughout space $L^2(\Gamma_e)$ and $L^2(\Gamma - (\Gamma_e \cup \Gamma_0))$, respectively, when v runs throughout space $\mathcal{K}(D)$, we see that

$$\Delta_g u = 0 \text{ on } \Gamma - (\Gamma_e \cup \Gamma_0), \quad \text{and} \quad \frac{\partial(\Delta_g u)}{\partial \nu} - \frac{1}{\mu^{\star}} \varrho^3 u = 0 \text{ on } \Gamma_e.$$

Therefore, (3.18) holds. In a similar way, we can prove the desired result for $G_{\mathcal{N}}$. \square

Theorem 3.7. *Let (\mathcal{M}, g) be a real analytic Riemannian manifold, and let $D \subset (\mathcal{M}, g)$ be a bounded domain with piecewise smooth boundary Γ . Let $D \in \mathcal{F}$. Assume that Γ_0 is a $C^{2,\varepsilon}$ -smooth $(n-1)$ -dimensional surface in $\Gamma - \bar{\Gamma}_\varrho$. If u is an eigenfunction of the transformations $G_{\mathcal{K}^d}^{(*)}$ with eigenvalue $\mu^* \neq 0$, then u has derivatives of any order in D and is such that*

$$(3.20) \quad \begin{cases} \Delta_g^2 u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_0, \\ \Delta_g u = 0, \quad \frac{\partial(\Delta_g u)}{\partial \nu} = 0 & \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma_0), \\ \frac{\partial(\Delta_g u)}{\partial \nu} - \lambda^3 \varrho^3 u = 0 & \text{on } \Gamma_\varrho, \quad \text{with } \lambda^3 = \frac{1}{\mu^*}. \end{cases}$$

Proof. If $G_{\mathcal{K}^d}^{(*)}u = \mu^*u$, then we have that $\frac{\partial u}{\partial \nu} = 0$ on Γ_ϱ and $u = \frac{\partial u}{\partial \nu} = 0$ on Γ_0 , and that

$$(3.21) \quad \int_{\Gamma_\varrho} \varrho^3 uv \, ds = \mu^* \int_D (\Delta_g u)(\Delta_g v) \, dR \quad \text{for all } v \in \mathcal{K}^d(D),$$

By using Green's formula and noticing that $\frac{\partial v}{\partial \nu}|_{\Gamma_\varrho} = 0$ and $v|_{\Gamma_0} = \frac{\partial v}{\partial \nu}|_{\Gamma_0} = 0$, we get that

$$(3.22) \quad \begin{aligned} \int_D (\Delta_g^2 u) v \, dR + \int_{\Gamma_\varrho} \left(\frac{\partial(\Delta_g u)}{\partial \nu} - \frac{1}{\mu^*} \varrho^3 u \right) v \, ds + \int_{\Gamma - (\Gamma_\varrho \cup \Gamma_0)} \frac{\partial(\Delta_g u)}{\partial \nu} v \, ds \\ - \int_{\Gamma - (\Gamma_\varrho \cup \Gamma_0)} (\Delta_g u) \frac{\partial v}{\partial \nu} \, ds = 0 \quad \text{for all } v \in K^d(D), \end{aligned}$$

where $\frac{\partial(\Delta_g u)}{\partial \nu} \in H^{-\frac{3}{2}}(\Gamma - (\Gamma_\varrho \cup \Gamma_0))$. By taking all $v \in C_0^\infty(D)$, we obtain that $\Delta_g^2 u = 0$ in D . Note that $\frac{\partial v}{\partial \nu}|_{\Gamma - (\Gamma_\varrho \cup \Gamma_0)}$ and $v|_{\Gamma - \Gamma_0}$ run throughout the spaces $L^2(\Gamma - (\Gamma_\varrho \cup \Gamma_0))$ and $L^2(\Gamma - \Gamma_0)$, respectively, when v runs throughout the space $K^d(D)$. Thus we have

$$\begin{aligned} \Delta_g u = 0 \quad \text{and} \quad \frac{\partial(\Delta_g u)}{\partial \nu} = 0 \quad \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma_0), \\ \text{and} \quad \frac{\partial(\Delta_g u)}{\partial \nu} - \frac{1}{\mu^*} \varrho^3 u = 0 \quad \text{on } \Gamma_\varrho. \end{aligned} \quad \square$$

Theorem 3.8. *Let (\mathcal{M}, g) , D , Γ_ϱ and Γ_0 be as in Theorem 3.7. Assume that γ_k^3 and κ_k^3 are the k -th Steklov eigenvalues of the following problems:*

$$(3.23) \quad \begin{cases} \Delta_g^2 u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{and} \quad \frac{\partial(\Delta_g u)}{\partial \nu} = 0 & \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma_0), \\ \frac{\partial(\Delta_g u)}{\partial \nu} - \varsigma^3 \varrho^3 u = 0 & \text{on } \Gamma_\varrho \end{cases}$$

and

$$\begin{cases} \Delta_g^2 u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_0, \\ \Delta_g u = 0 \quad \text{and} \quad \frac{\partial(\Delta_g u)}{\partial \nu} = 0 & \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma_0), \\ \frac{\partial(\Delta_g u)}{\partial \nu} - \kappa^3 \varrho^3 u = 0 & \text{on } \Gamma_\varrho, \end{cases}$$

respectively. Then $\varsigma_k^3 \leq \kappa_k^3$ for all $k \geq 1$.

Proof. For $0 \leq \alpha \leq 1$, let $u_k = u_k(\alpha, x)$ be the normalized eigenfunction corresponding to the k -th Steklov eigenvalue λ_k for the following problem:

$$(3.24) \quad \begin{cases} \Delta_g^2 u_k = 0 & \text{in } D, \\ \frac{\partial u_k}{\partial \nu} = 0 & \text{on } \Gamma_\varrho, \\ u_k = \frac{\partial u_k}{\partial \nu} = 0 & \text{on } \Gamma_0, \\ \alpha \Delta_g u_k + (1 - \alpha) \frac{\partial u_k}{\partial \nu} = 0 \text{ and } \frac{\partial(\Delta_g u_k)}{\partial \nu} = 0 & \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma_0), \\ \frac{\partial(\Delta_g u_k)}{\partial \nu} - \lambda \varrho^3 u_k = 0 & \text{on } \Gamma_\varrho. \end{cases}$$

It is easy to verify (cf, p. 410 or Theorem 9 of p. 419 in [5]) that the k -th Steklov eigenvalue $\lambda_k = \lambda_k(\alpha)$ is continuous on the closed interval $[0, 1]$ and differentiable in the open interval $(0, 1)$, and that $u_k(\alpha, x)$ is also differentiable with respect to α in $(0, 1)$ (cf. [8]). We will denote by $'$ the derivative with respect to α . Then

$$(3.25) \quad \begin{cases} \Delta_g^2 u'_k = 0 & \text{in } D, \\ \frac{\partial u'_k}{\partial \nu} = 0 & \text{on } \Gamma_\varrho, \\ u'_k = \frac{\partial u'_k}{\partial \nu} = 0 & \text{on } \Gamma_0, \\ \Delta_g u_k + \alpha \Delta_g u'_k - \frac{\partial u_k}{\partial \nu} + (1 - \alpha) \frac{\partial u'_k}{\partial \nu} = 0 \text{ and } \frac{\partial(\Delta_g u'_k)}{\partial \nu} = 0 & \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma_0) \\ \frac{\partial(\Delta_g u'_k)}{\partial \nu} - \lambda' \varrho^3 \frac{\partial u_k}{\partial \nu} - \lambda \varrho^3 \frac{\partial u'_k}{\partial \nu} = 0 & \text{on } \Gamma_\varrho. \end{cases}$$

Multiplying (3.25) by u_k , integrating the product over D , and then applying Green's formula, we get that for $0 < \alpha < 1$

$$\begin{aligned} 0 &= \int_D (\Delta_g^2 u'_k) u_k dR = \int_D (\Delta_g^2 u_k) u'_k dR - \int_{\partial D} (\Delta_g u_k) \frac{\partial u'_k}{\partial \nu} ds \\ &\quad + \int_{\partial D} u'_k \frac{\partial(\Delta_g u_k)}{\partial \nu} ds - \int_{\partial D} u_k \frac{\partial(\Delta_g u'_k)}{\partial \nu} ds + \int_{\partial D} (\Delta_g u'_k) \frac{\partial u_k}{\partial \nu} ds \\ &= \left[\int_{\Gamma_\varrho} u'_k \frac{\partial(\Delta_g u_k)}{\partial \nu} ds - \int_{\Gamma - (\Gamma_\varrho \cup \Gamma_0)} (\Delta_g u_k) \frac{\partial u'_k}{\partial \nu} ds \right] \\ &\quad + \left[- \int_{\Gamma_\varrho} u_k \frac{\partial(\Delta_g u'_k)}{\partial \nu} ds + \int_{\Gamma - (\Gamma_\varrho \cup \Gamma_0)} (\Delta_g u'_k) \frac{\partial u_k}{\partial \nu} ds \right] \\ &= \left[\int_{\Gamma_\varrho} \lambda \varrho^3 u_k u'_k ds + \int_{\Gamma - (\Gamma_\varrho \cup \Gamma_0)} \left(\frac{1 - \alpha}{\alpha} \frac{\partial u_k}{\partial \nu} \right) \frac{\partial u'_k}{\partial \nu} ds \right] \\ &\quad + \int_{\Gamma_\varrho} (-\lambda' \varrho^3 u_k - \lambda \varrho^3 u'_k) u_k ds \\ &\quad + \int_{\Gamma - (\Gamma_\varrho \cup \Gamma_0)} \left(-\frac{1}{\alpha} \Delta_g u_k + \frac{1}{\alpha} \frac{\partial u_k}{\partial \nu} - \frac{1 - \alpha}{\alpha} \frac{\partial u'_k}{\partial \nu} \right) \frac{\partial u_k}{\partial \nu} ds \\ &= -\lambda' \int_{\Gamma_\varrho} \varrho^3 u_k^2 ds + \int_{\Gamma - (\Gamma_\varrho \cup \Gamma_0)} \left[\left(\frac{1 - \alpha}{\alpha^2} \right) \frac{\partial u_k}{\partial \nu} + \frac{1}{\alpha} \frac{\partial u_k}{\partial \nu} \right] \frac{\partial u_k}{\partial \nu} ds \\ &= -\lambda' \int_{\Gamma_\varrho} \varrho^2 u_k^2 ds + \int_{\Gamma - (\Gamma_\varrho \cup \Gamma_0)} \left(\frac{1}{\alpha} \frac{\partial u_k}{\partial \nu} \right)^2 ds, \end{aligned}$$

i.e.,

$$\lambda'_k(\alpha) = \frac{\int_{\Gamma - (\Gamma_\varrho \cup \Gamma_0)} \left(\frac{1}{\alpha} \frac{\partial u_k}{\partial \nu} \right)^2 ds}{\int_{\Gamma_\varrho} \varrho^3 u_k^2 ds} > 0 \quad \text{for all } 0 < \alpha < 1.$$

This implies that λ_k is increasing with respect to α in $(0, 1)$. Note that if we change the α from 0 to 1, each individual Steklov eigenvalue λ_k increase monotonically from the value ς_k which is the k -th Steklov eigenvalue of (3.20) to the value κ_k which is the k -th Steklov eigenvalue (3.24). Thus, we have that $\varsigma_k \leq \kappa_k$ for all k . \square

Conversely, we can show that a sufficiently smooth function satisfying (3.18) (respectively, (3.20)) is an eigenfunction of $G_{\mathcal{K}}^{(*)}$ or $G_{\mathcal{N}}^{(*)}$ (respectively, $G_{\mathcal{K}^d}^{(*)}$).

Proposition 3.9. *Let \bar{D} be bounded domain with piecewise smooth boundary, and let $D \in \mathcal{F}$. Assume that u belongs to $C^4(\bar{D})$ and let $\lambda > 0$.*

a) *If $\Gamma_\varrho \neq \Gamma$ and u satisfies (3.18), then $u \in \mathcal{K}$ and u is an eigenfunction of $G_{\mathcal{K}}^{(*)}$ with the eigenvalue $\mu^* = \lambda^{-3}$,*

$$(3.26) \quad G_{\mathcal{K}}^{(*)} u = \lambda^{-3} u.$$

b) *If $\Gamma_\varrho \neq \Gamma$ and u satisfies (3.20), then $u \in \mathcal{K}^d$ and u is an eigenfunction of $G_{\mathcal{K}^d}^{(*)}$ with the eigenvalue $\mu^* = \lambda^{-3}$,*

$$(3.27) \quad G_{\mathcal{K}^d}^{(*)} u = \lambda^{-3} u.$$

c) *If $\Gamma_\varrho = \Gamma$ and u satisfies (3.18), then $u \in \mathcal{N}$ and u is an eigenfunction of $G_{\mathcal{N}}^{(*)}$ with the eigenvalue $\mu^* = \lambda^{-3}$,*

$$(3.28) \quad G_{\mathcal{N}}^{(*)} u = \lambda^{-3} u.$$

Proof. i) $\Gamma_\varrho \neq \Gamma$. We claim that there is no eigenvalue $\lambda^3 = 0$. Suppose by contradiction that there is a function u in $C^4(\bar{D})$ satisfying

$$(3.29) \quad \begin{cases} \Delta_g^2 u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho \cup \Gamma_0, \quad u = 0 & \text{on } \Gamma - \Gamma_\varrho \\ \Delta_g u = 0 & \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma_0), \quad \text{and } \frac{\partial(\Delta_g u)}{\partial \nu} = 0 & \text{on } \Gamma_\varrho. \end{cases}$$

Multiplying the above equation by u , integrating the result over D , and then using Green's formula, we derive

$$\begin{aligned} 0 &= \int_D u(\Delta_g^2 u) dR = \int_D |\Delta_g u|^2 dR - \int_\Gamma u \frac{\partial(\Delta_g u)}{\partial \nu} ds \\ &\quad + \int_\Gamma (\Delta_g u) \frac{\partial u}{\partial \nu} ds = \int_D |\Delta_g u|^2 dR. \end{aligned}$$

This implies that $\Delta_g u = 0$ in D , so that

$$0 = \int_D u(\Delta_g u) dR = - \int_D |\nabla u|^2 dR - \int_{\partial D} u \frac{\partial u}{\partial \nu} ds = - \int_D |\nabla u|^2 dR.$$

That is, $u \equiv \text{constant}$ in D . Since $u = 0$ on $\Gamma - \Gamma_\varrho$, we get that $u = 0$ in D . The claim is proved.

In view of assumptions, we see that $u \in \mathcal{K}$. By (3.18) and Green's formula, it follows that for an arbitrary $v \in K(D)$

$$\begin{aligned}
\langle G_{\mathcal{K}}^{(*)}u, v \rangle^* &= [u, v] = \int_{\Gamma_e} \varrho^3 uv \, ds \\
&= \lambda^{-3} \int_{\Gamma_e} \frac{\partial(\Delta_g u)}{\partial \nu} v \, ds = \lambda^{-3} \int_{\Gamma} \frac{\partial(\Delta_g u)}{\partial \nu} v \, ds \\
&= \lambda^{-3} \left[\int_{\Gamma} (\Delta_g u) \frac{\partial v}{\partial \nu} \, ds + \int_D (\Delta_g u)(\Delta_g v) \, dR - \int_D v(\Delta_g^2 u) \, dR \right] \\
&= \lambda^{-3} \int_D (\Delta_g u)(\Delta_g v) \, dR = \lambda^{-3} \langle u, v \rangle^*,
\end{aligned}$$

Therefore,

$$\langle G_{\mathcal{K}}^{(*)}u - \lambda^{-3}u, v \rangle^* = 0 \quad \text{for all } v \in K(D),$$

which implies (3.26). By a similar way, we can prove b).

ii) $\Gamma_e = \Gamma$. In this case, for the eigenvalue $\lambda^3 = 0$, the problem (3.18) has the solutions $u_{01} = \text{constant}$ and $u_{02}(x) = \int_D F(x, y) \, dR_y$ in D , here $F(x, y)$ is Green's function with Neumann boundary condition (see, Section 1). These solutions do not belong to $N(D)$. If, however, u is a solution with eigenvalue $\lambda^3 > 0$ then $u \in \mathcal{N}$. Indeed, by Green's formula we get

$$\int_{\partial D} \frac{\partial(\Delta_g u)}{\partial \nu} \, ds = \int_D \Delta_g^2 u \, dR = 0$$

and hence from (3.18) we obtain

$$\int_{\partial D} \varrho^3 u u_{01} \, ds = \int_{\partial D} \varrho^3 u \, ds = 0.$$

In addition, from (3.18) we get

$$\begin{cases} \Delta_g(\Delta_g u) = 0 & \text{in } D, \\ \frac{\partial(\Delta_g u)}{\partial \nu} = \lambda^3 \varrho^3 u & \text{on } \partial D. \end{cases}$$

so that

$$\Delta_g u(x) = \int_{\partial D} F(x, y) \frac{\partial(\Delta_g u)}{\partial \nu_y} \, ds_y = \lambda^3 \int_{\partial D} F(x, y) \varrho^3(y) u(y) \, ds_y.$$

Combining this and Green's formula, we have

$$\begin{aligned}
0 &= - \int_{\partial D} \frac{\partial u}{\partial \nu} \, ds = \int_D \Delta_g u \, dR = \lambda^3 \int_D \left(\int_{\partial D} F(x, y) \varrho^3(y) u(y) \, ds_y \right) \, dR_x \\
&= \lambda^3 \int_{\partial D} \varrho^3(y) u(y) \left(\int_D F(x, y) \, dR_x \right) \, ds_y \\
&= \lambda^3 \int_{\partial D} \varrho^3(x) u(x) \left(\int_D F(x, y) \, dR_y \right) \, ds_x = \lambda^3 \int_{\partial D} \varrho^3(x) u(x) u_{02}(x) \, ds_x,
\end{aligned}$$

i.e., $\int_{\partial D} \varrho^3 u_{02} u \, ds = 0$, so that $u \in \mathcal{N}$. Proceeding as in a), we can prove that (3.28) holds. \square

Remark 3.10. Each of transformations $G_{\mathcal{K}}^*$, $G_{\mathcal{K}^d}^*$ and $G_{\mathcal{N}}^*$ corresponds to a biharmonic Steklov problem given by the quadratic forms

$$\langle u, u \rangle^* = \int_D |\Delta_g u|^2 dR$$

and

$$[u, u] = \int_{\Gamma_\varrho} \varrho^3 u^2 ds$$

and the function classes of \mathcal{K}^* , \mathcal{K}^{d*} and \mathcal{N}^* , respectively. The eigenvalues λ_k^3 of these biharmonic Steklov problems are given by

$$(3.30) \quad \lambda_k^3 = 1/\mu_k^*, \quad k = 1, 2, 3, \dots$$

Since 0 is the only limit point of μ_k^* , the only possible limit points of λ_k^3 are $+\infty$.

4. BIHARMONIC STEKLOV EIGENVALUES ON A RECTANGULAR PARALLELEPIPED

Let $D = \{x \in \mathbb{R}^n | 0 \leq x_i \leq l_i, i = 1, \dots, n\}$ with boundary Γ , and let $\Gamma_\varrho = \{x \in \mathbb{R}^n | 0 \leq x_i \leq l_i \text{ when } i < n, x_n = 0\}$. Let $\Gamma^{l_n} = \{x \in \mathbb{R}^n | 0 \leq x_i \leq l_i \text{ when } i < n, x_n = l_n\}$. Our first purpose, in this section, is to discuss the biharmonic Steklov eigenvalue problem on n -dimensional rectangular parallelepiped D :

$$(4.1) \quad \begin{cases} \Delta^2 u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma^{l_n}, \\ u = \Delta u = 0 & \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma^{l_n}), \\ \frac{\partial(\Delta u)}{\partial \nu} - \lambda^3 \varrho^3 u = 0 & \text{on } \Gamma_\varrho, \quad \varrho = \text{constant} > 0 \text{ on } \Gamma_\varrho. \end{cases}$$

We consider nonzero product solution of (4.1) of the form:

$$u = X(x_1, \dots, x_{n-1}) Y(x_n),$$

where $X(x_1, \dots, x_{n-1})$ is a function of variables x_1, \dots, x_{n-1} and $Y(x_n)$ is a function of x_n alone. Since

$$\begin{aligned} \Delta u &= (\Delta_{n-1} X(x_1, \dots, x_{n-1})) Y(x_n) + 2 \nabla X(x_1, \dots, x_{n-1}) \cdot \nabla Y(x_n) \\ &\quad + (X(x_1, \dots, x_{n-1})) Y''(x_n) = (\Delta_{n-1} X(x_1, \dots, x_{n-1})) Y(x_n) \\ &\quad + (X(x_1, \dots, x_{n-1})) Y''(x_n) \end{aligned}$$

and

$$\begin{aligned} \Delta^2 u &= (\Delta_{n-1}^2 X(x_1, \dots, x_{n-1})) Y(x_n) + 2 (\Delta_{n-1} X(x_1, \dots, x_{n-1})) Y''(x_n) \\ &\quad + (X(x_1, \dots, x_{n-1})) Y''''(x_n), \end{aligned}$$

where

$$\Delta_{n-1} X(x_1, \dots, x_{n-1}) = \sum_{i=1}^{n-1} \frac{\partial^2 X}{\partial x_i^2},$$

we find by $\Delta^2 u = 0$ that

$$\begin{aligned} & (\Delta^2 X(x_1, \dots, x_{n-1}))Y(x_n) + 2(\Delta X(x_1, \dots, x_{n-1}))Y''(x_n) \\ & + (X(x_1, \dots, x_{n-1}))Y''''(x_n) = 0, \end{aligned}$$

so that

$$(4.2) \quad \frac{\Delta^2 X(x_1, \dots, x_{n-1})}{X(x_1, \dots, x_{n-1})} + 2 \frac{\Delta X(x_1, \dots, x_{n-1})}{X(x_1, \dots, x_{n-1})} \frac{Y''(x_n)}{Y(x_n)} + \frac{Y''''(x_n)}{Y(x_n)} = 0.$$

Differentiating (4.2) with respect to x_n , we obtain that

$$2 \frac{\Delta X(x_1, \dots, x_{n-1})}{X(x_1, \dots, x_{n-1})} \left[\frac{Y''(x_n)}{Y(x_n)} \right]' + \left[\frac{Y''''(x_n)}{Y(x_n)} \right]' = 0.$$

The above equation holds if and only if

$$(4.3) \quad \frac{\Delta X(x_1, \dots, x_{n-1})}{X(x_1, \dots, x_{n-1})} = - \frac{\left[\frac{Y''''(x_n)}{Y(x_n)} \right]'}{2 \left[\frac{Y''(x_n)}{Y(x_n)} \right]'} = -\eta^2,$$

where η^2 is a constant. Therefore, we have that

$$(4.4) \quad \Delta X(x_1, \dots, x_{n-1}) + \eta^2 X(x_1, \dots, x_{n-1}) = 0$$

and

$$\left[\frac{Y''''(x_n)}{Y(x_n)} \right]' - 2\eta^2 \left[\frac{Y''(x_n)}{Y(x_n)} \right]' = 0.$$

From (4.4), we get

$$(4.5) \quad \Delta^2 X = -\eta^2 \Delta X = \eta^4 X.$$

Substituting this into (4.2), we obtain the following equation

$$(4.6) \quad Y''''(x_n) - 2\eta^2 Y''(x_n) + \eta^4 Y(x_n) = 0.$$

It is easy to verify that the general solutions of (4.6) have the form:

$$(4.7) \quad Y(x_n) = A \cosh \eta x_n + B \sinh \eta x_n + C x_n \cosh \eta x_n + D x_n \sinh \eta x_n.$$

By setting $Y(0) = 1$, $Y(l_n) = 0$, $Y'(0) = 0$, $Y'(l_n) = 0$, we get

$$(4.8) \quad \begin{aligned} Y(x_n) &= \cosh \eta x_n - \left[\frac{(\sinh \eta l_n)(\cosh \eta l_n) + \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right] \sinh \eta x_n \\ &+ \left[\frac{\eta(\sinh \eta l_n)(\cosh \eta l_n) + \eta^2 l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right] x_n \cosh \eta x_n \\ &- \left[\frac{\eta \sinh^2 \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right] x_n \sinh \eta x_n. \end{aligned}$$

It is well-known that for the Dirichlet eigenvalue problem

$$(4.9) \quad \begin{cases} \Delta X(x_1, \dots, x_{n-1}) + \eta^2 X(x_1, \dots, x_{n-1}) = 0 & \text{in } D, \\ u = 0 & \text{on } \partial\{(x_1, \dots, x_{n-1}) | 0 \leq x_i \leq l_i, i = 1, \dots, n-1\}, \end{cases}$$

there exist the eigenfunctions

$$(4.10) \quad X(x_1, \dots, x_{n-1}) = c \left(\sin \frac{m_1 \pi}{l_1} x_1 \right) \cdots \left(\sin \frac{m_{n-1} \pi}{l_{n-1}} x_{n-1} \right),$$

which correspond to the eigenvalues

$$\eta^2 = \sum_{i=1}^{n-1} \left(\frac{m_i \pi}{l_i} \right)^2, \quad \text{where } m_i = 1, 2, 3, \dots$$

Therefore,

$$\begin{aligned} (4.11) \quad u &= (X(x_1, \dots, x_{n-1}))Y(x_n) \\ &= c \left(\sin \frac{m_1 \pi}{l_1} x_1 \right) \cdots \left(\sin \frac{m_{n-1} \pi}{l_{n-1}} x_{n-1} \right) [\cosh \eta x_n \\ &\quad - \left(\frac{(\sinh \eta l_n)(\cosh \eta l_n) + \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) \sinh \eta x_n \\ &\quad + \left(\frac{\eta (\sinh \eta l_n)(\cosh \eta l_n) + \eta^2 l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) x_n \cosh \eta x_n \\ &\quad - \left(\frac{\eta \sinh^2 \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) x_n \sinh \eta x_n]. \end{aligned}$$

Since

$$Y'''(0) = 2\eta^3 \left(\frac{(\sinh \eta l_n)(\cosh \eta l_n) + \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right),$$

we obtain

$$\begin{aligned} \frac{\partial(\Delta u)}{\partial \nu} \Big|_{x_n=0} &= (\Delta_{n-1} X(x_1, \dots, x_{n-1}))Y'(0) + (X(x_1, \dots, x_{n-1}))Y'''(0) \\ &= 2\eta^3 \left(\frac{(\sinh \eta l_n)(\cosh \eta l_n) + \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) X(x_1, \dots, x_{n-1}), \end{aligned}$$

so that

$$\frac{\partial(\Delta u)}{\partial \nu} - \lambda^3 \varrho^3 u = 0 \quad \text{on } \Gamma_\varrho$$

with

$$\lambda^3 = \frac{2\eta^3 l_n^3}{\rho^3 l_n^3} \left(\frac{(\sinh \eta l_n)(\cosh \eta l_n) + l_n \eta}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right).$$

Our second purpose is to discuss the biharmonic Steklov eigenvalue problem on the n -dimensional rectangular parallelepiped D :

$$(4.12) \quad \begin{cases} \Delta^2 u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho, \quad u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma^{l_n}, \\ \frac{\partial(\Delta u)}{\partial \nu} = 0 & \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma^{l_n}), \\ \frac{\partial(\Delta u)}{\partial \nu} - \lambda^3 \varrho^3 u = 0 & \text{on } \Gamma_\varrho, \quad \varrho = \text{constant} > 0 & \text{on } \Gamma_\varrho. \end{cases}$$

Similarly, (4.12) has the special solution $u = (X(x_1, \dots, x_{n-1}))Z(x_n)$ with $Z(x_n)$ having form (4.7). According to the boundary conditions of (4.12), we get that the problem (4.12) has the solutions

$$\begin{aligned} u(x) &= u(x_1, \dots, x_n) \\ &= c \left(\cos \frac{m_1 \pi}{l_1} x_1 \right) \cdots \left(\cos \frac{m_{n-1} \pi}{l_{n-1}} x_{n-1} \right) Z(x_n), \end{aligned}$$

where m_1, \dots, m_{n-1} are whole numbers, and $Z(x_n)$ is given by

$$(4.13) \quad \begin{aligned} Z(x_n) = & \cosh \beta x_n - \left[\frac{(\sinh \beta l_n)(\cosh \beta l_n) + \beta l_n}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right] \sinh \beta x_n \\ & + \left[\frac{\beta(\sinh \beta l_n)(\cosh \beta l_n) + \beta^2 l_n}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right] x_n \cosh \beta x_n \\ & - \left[\frac{\beta \sinh^2 \beta l_n}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right] x_n \sinh \beta x_n, \end{aligned}$$

$\beta = [\sum_{i=1}^{n-1} (m_i \pi / l_i)^2]^{1/2}$ with $\sum_{i=1}^{n-1} m_i \neq 0$. Since $Z'''(0) = 2\beta^3 \left(\frac{(\sinh \beta l_n)(\cosh \beta l_n) + \beta l_n}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right)$ and $\frac{\partial(\Delta u)}{\partial \nu} \Big|_{x_n=0} = (X(x_1, \dots, x_{n-1})) Z'''(0)$, we get $\frac{\partial(\Delta u)}{\partial \nu} - \lambda^3 \varrho^3 u = 0$ on Γ_ϱ , where

$$\lambda^3 = \frac{2\beta^3 l_n^3}{\rho^3 l_n^3} \left(\frac{(\sinh \beta l_n)(\cosh \beta l_n) + \beta l_n}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right).$$

5. ASYMPTOTIC DISTRIBUTION OF EIGENVALUES ON SPECIAL DOMAINS

5.1. Counting function $A(\tau)$.

In order to obtain our asymptotic formula, it is an effective way to investigate the distribution of the eigenvalues of the transformation G_K (respectively, G_{K^d} , G_N) instead of the transformations $G_K^{(*)}$ (respectively, $G_{K^d}^{(*)}$, $G_N^{(*)}$). It follows from (3.12), (3.13), (3.14) and (3.30) we obtain

$$(5.1) \quad \mu_k = (1 + \lambda_k^3)^{-1}, \quad k = 1, 2, 3, \dots,$$

where μ_k denote the k -th eigenvalue of G_K or G_{K^d} or G_N , and $\frac{1}{\lambda_k}$ is the k -th eigenvalue of $G_K^{(*)}$ or $G_{K^d}^{(*)}$ or $G_N^{(*)}$. Since $A(\tau) = \sum_{\lambda_k \leq \tau} 1$, we have

$$(5.2) \quad A(\tau) = \sum_{\mu_k \geq (1+\tau^3)^{-1}} 1.$$

5.2. D is an n -dimensional rectangular parallelepiped and $g^{ik} = \delta^{ik}$.

Let D be an n -dimensional rectangular parallelepiped, $g^{ik} = \delta^{ik}$ in the whole of \bar{D} , $\varrho = \text{constant} > 0$ on one face Γ_ϱ^+ of the rectangular parallelepiped and $\varrho = 0$ on $\Gamma_\varrho - \Gamma_\varrho^+$, i.e., $D = \{x \in \mathbb{R}^n | 0 \leq x_i \leq l_i, i = 1, \dots, n\}$, $\Gamma_\varrho^+ = \{x \in \mathbb{R}^n | 0 \leq x_i \leq l_i \text{ when } i < n, x_n = 0\}$ and $\Gamma_0 = \Gamma^{l_n} = \{x \in \mathbb{R}^n | 0 \leq x_i \leq l_i \text{ when } i < n, x_n = l_n\}$. Without loss of generality, we assume $l_i < l_n$ for all $i < n$.

For the above domain D , except for the $K(D)$ and $K^d(D)$ in Section 3, we introduce the linear space of functions

$$K^0(D) = \{u | u \in Lip(\bar{D}) \cap H^2(D), \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_\varrho \cup \Gamma_0, u = 0 \text{ on } \Gamma - \Gamma_\varrho^+\},$$

Clearly,

$$(5.3) \quad K^0(D) \subset K(D) \subset K^d(D),$$

Closing K^0 , K and K^d respect to the norm $\|u\| = \sqrt{\langle u, u \rangle}$, we obtain the Hilbert spaces \mathcal{K}^0 , \mathcal{K} and \mathcal{K}^d , and

$$(5.4) \quad \mathcal{K}^0 \subset \mathcal{K} \subset \mathcal{K}^d.$$

According to Theorem 3.3, we see that the bilinear functional

$$(5.5) \quad [u, v] = \int_{\Gamma_\varrho^+} \varrho^3 uv \, ds$$

defines self-adjoint, completely continuous transformations G^0 , G and G^d on \mathcal{K}^0 , \mathcal{K} and \mathcal{K}^d , respectively (cf. Section 3). Obviously,

$$\langle G^0 u, v \rangle = \langle Gu, v \rangle \quad \text{for all } u, v \text{ in } \mathcal{K}^0,$$

$$\langle Gu, v \rangle = \langle G^d u, v \rangle \quad \text{for all } u, v \text{ in } \mathcal{K},$$

from which and applying Theorem 1.4 of [26] we immediately get

$$(5.6) \quad \mu_k^0 \leq \mu_k \leq \mu_k^d, \quad k = 1, 2, 3, \dots,$$

where $\{\mu_k^0\}$ and $\{\mu_k^d\}$ are the eigenvalues of G^0 and G^d , respectively. Hence

$$(5.7) \quad A^0(\tau) \leq A(\tau) \leq A^d(\tau) \quad \text{for all } \tau,$$

where

$$(5.8) \quad A^0(\tau) = \sum_{\mu_k^0 \geq (1+\tau^3)^{-1}} 1$$

and

$$(5.9) \quad A^d(\tau) = \sum_{\mu_k^d \geq (1+\tau^3)^{-1}} 1.$$

We shall estimate the asymptotic behavior of $A^0(\tau)$ and $A^d(\tau)$. It is easy to verify (cf. Theorems 3.6, 3.7) that the eigenfunctions of the transformations G^0 and G^d , respectively, satisfy

$$(5.10) \quad \begin{cases} \Delta^2 u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho^+, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma^{l_n}, \text{ and } u = \Delta u = 0 \text{ on } \Gamma - (\Gamma_\varrho^+ \cup \Gamma^{l_n}), \\ \frac{\partial(\Delta u)}{\partial \nu} - \lambda^3 \varrho^3 u = 0 & \text{on } \Gamma_\varrho^+, \quad \varrho = \text{constant} > 0 \text{ on } \Gamma_\varrho^+. \end{cases}$$

and

$$(5.11) \quad \begin{cases} \Delta^2 u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho^+, \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma^{l_n}, \\ \frac{\partial(\Delta u)}{\partial \nu} = 0 \text{ and } \Delta u = 0 & \text{on } \Gamma - (\Gamma_\varrho^+ \cup \Gamma^{l_n}), \\ \frac{\partial(\Delta u)}{\partial \nu} - \kappa^3 \varrho^3 u = 0 & \text{on } \Gamma_\varrho^+, \quad \varrho = \text{constant} > 0 \text{ on } \Gamma_\varrho^+. \end{cases}$$

As being verified in Section 4, the functions of form

$$(5.12) \quad u(x) = c \left(\sin \frac{m_1 \pi}{l_1} x_l \right) \cdots \left(\sin \frac{m_{n-1} \pi}{l_{n-1}} x_{n-1} \right) Y(x_n)$$

are the solutions of the problem (5.10), where $Y(x_n)$ is given by (4.8). Since the functions in (5.12) have derivatives of any order in D , it follows from Proposition 3.9 that they are eigenfunctions of the transformation G^0 with eigenvalues $(1 + \lambda^3)^{-1}$, where

$$(5.13) \quad \lambda^3 = \frac{2\eta^3 l_n^3}{\varrho^3 l_n^3} \left(\frac{(\sinh \eta l_n)(\cosh \eta l_n) + \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right),$$

$$\text{and } \eta = \left[\sum_{i=1}^{n-1} \left(\frac{m_i \pi}{l_i} \right)^2 \right]^{1/2}, \quad i = 1, 2, 3, \dots.$$

Note that the restriction of u on Γ_ϱ

$$(5.14) \quad u|_{\Gamma_\varrho} = c \left(\sin \frac{m_1 \pi}{l_1} x_1 \right) \cdots \left(\sin \frac{m_{n-1} \pi}{l_{n-1}} x_{n-1} \right),$$

when m_1, \dots, m_{n-1} run through all positive integers (see, Section 4), form a complete system of orthogonal functions in $L_\varrho^2(\Gamma_\varrho)$. It follows from Proposition 3.5 that if m_1, \dots, m_{n-1} run through all positive integers, then the functions (5.12) form an orthogonal basis of the subspace of \mathcal{K}^0 , spanned by the eigenfunctions of G^0 , corresponding to positive eigenvalues. That is, when m_1, \dots, m_{n-1} run through all positive integers, then $(1 + \lambda^3)^{-1}$, where λ^3 is given by (5.13), runs through all positive eigenvalues of G^0 .

Similarly, for the problem (5.11), the eigenfunctions $\{u_k\}$ of the operator G^d on \mathcal{K}^d , corresponding to non-zero eigenvalues, form an orthogonal basis of the subspace of \mathcal{K}^d . The non-zero eigenvalues of G^d are $\mu_k^d = (1 + \kappa_k^3)^{-1}$, where κ_k^3 is the k -th Steklov eigenvalue of (5.11).

In order to give the upper bound estimate of $A^d(\tau)$, we further introduce the following Steklov eigenvalue problem

$$(5.15) \quad \begin{cases} \Delta^2 u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho^+, \quad u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma^{l_n}, \\ \frac{\partial u}{\partial \nu} = \frac{\partial(\Delta u)}{\partial \nu} = 0 & \text{on } \Gamma - (\Gamma_\varrho^+ \cup \Gamma^{l_n}), \\ \frac{\partial(\Delta u)}{\partial \nu} - \gamma^3 \varrho^3 u = 0 & \text{on } \Gamma_\varrho^+, \quad \varrho = \text{constant} > 0 & \text{on } \Gamma_\varrho^+. \end{cases}$$

Let γ_k^3 be the k -th eigenvalue of (5.15). By Theorem 3.8, we have

$$(5.16) \quad \gamma_k^3 \leq \kappa_k^3 \quad \text{for all } k \geq 1.$$

We define

$$(5.17) \quad \mu_k^f = \frac{1}{1 + \gamma_k^3}, \quad A^f(\tau) = \sum_{\mu_k^f \geq (1 + \tau^3)^{-1}} 1.$$

It follows from (5.16) and (5.17) that

$$(5.18) \quad A^d(\tau) \leq A^f(\tau) \quad \text{for all } \tau.$$

We know (cf. Section 4) that the problem (5.15) has the solutions of form

$$(5.19) \quad u(x) = c \left(\cos \frac{m_1 \pi}{l_1} x_1 \right) \cdots \left(\cos \frac{m_{n-1} \pi}{l_{n-1}} x_{n-1} \right) Z(x_n),$$

where m_1, \dots, m_{n-1} are non-negative integers with $\sum_{i=1}^{n-1} m_i \neq 0$, and $Z(x_n)$ is given by (4.13). This implies that if m_1, \dots, m_{n-1} run through all non-negative integers with

$\sum_{i=1}^{n-1} m_i \neq 0$, then

$$(5.20) \quad \gamma^3 = \frac{2\beta^3 l_n^3}{\rho^3 l_n^3} \left(\frac{(\sinh \beta l_n)(\cosh \beta l_n) + \beta l_n}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right), \quad \beta = \left[\sum_{i=1}^{n-1} \left(\frac{m_i \pi}{l_i} \right)^2 \right]^{1/2}$$

runs throughout all eigenvalues of problem (5.15).

We first compute the asymptotic behavior of $A^f(\tau)$. By (5.17), (5.20) and the argument as in p.44 of [37] or p.373 of [5] or p.51-53 of [26], $A^f(\tau)$ = the number of $(n-1)$ -tuples (m_1, \dots, m_{n-1}) satisfying the inequality

$$(5.21) \quad \frac{2\beta^3 l_n^3}{\rho^3 l_n^3} \left(\frac{(\sinh \beta l_n)(\cosh \beta l_n) + \beta l_n}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right) \leq \tau^3,$$

where m_1, \dots, m_{n-1} are non-negative integers with $\sum_{i=1}^{n-1} m_i \neq 0$. By setting

$$(5.22) \quad t(s) = 2s^3 \left(\frac{(\sinh s)(\cosh s) + s}{\sinh^2 s - s^2} \right),$$

we see that

$$\lim_{s \rightarrow +\infty} t(s)/s^3 = 2.$$

We claim that for all $s \geq 1$,

$$t'(s) = \frac{2s^2 [-s^3 + 3s \sinh^2 s + 3(\sinh^3 s)(\cosh s) - 3s^2(\sinh s)(\cosh s) - 2s^3 \cosh^2 s]}{(\sinh^2 s - s^2)^2} > 0.$$

In fact, let

$$\theta(s) = -s^3 + 3s \sinh^2 s + 3(\sinh^3 s)(\cosh s) - 3s^2(\sinh s)(\cosh s) - 2s^3 \cosh^2 s.$$

Then

$$\begin{aligned} \theta(1) &> 0, \text{ and} \\ \theta'(s) &= 4(\cosh^2 s) \left[3 \sinh^2 s - 3s^2 - s^3 \frac{\sinh s}{\cosh s} \right] \\ &> 4(\cosh^2 s) [3 \sinh^2 s - 3s^2 - s^3] \\ &> 4(\cosh^2 s) \left[\frac{3}{4}(e^{2s} + e^{-2s}) - \frac{3}{2} - 3s^2 - s^3 \right] \\ &> 4(\cosh^2 s) \left[\frac{3}{4} \left(2 + 4s^2 + \frac{4}{3}s^4 \right) - \frac{3}{2} - 3s^2 - s^3 \right] \\ &= 4(\cosh^2 s)(s^4 - s^3) \geq 0 \quad \text{for } s \geq 1. \end{aligned}$$

This implies that $\theta(s) \geq 0$ for $s \geq 1$. Thus, the function $t(s)$ is increasing in $[1, +\infty)$. Denote by $s = h(t)$ the inverse of function $t(s)$ for $s \geq 1$. Then

$$\lim_{t \rightarrow +\infty} \frac{(h(t))^3}{t} = \frac{1}{2}.$$

Note that, for $s \geq 1$, the inequalities $t(s) \leq t$ is equivalent to $s \leq h(t)$. Hence (5.21) is equivalent to

$$\beta l_n \leq h(\rho^3 l_n^3 \tau^3),$$

which can be written as

$$(5.23) \quad \sum_{i=1}^{n-1} (m_i/l_i)^2 \leq \left[\frac{1}{\pi l_n} h(\varrho^3 l_n^3 \tau^3) \right]^2, \quad m_i = 0, 1, 2, \dots \text{ with } \sum_{i=1}^{n-1} m_i \neq 0.$$

We consider the $(n-1)$ -dimensional ellipsoid

$$\sum_{i=1}^{n-1} (z_i/l_i)^2 \leq \left[\frac{1}{\pi l_n} h(\varrho^3 l_n^3 \tau^3) \right]^2.$$

Since $A^f(\tau) + 1$ just is the number of those $(n-1)$ -dimensional unit cubes of the z -space that have corners whose coordinates are non-negative integers in the ellipsoid (see, VI. §4 of [5]). Hence $A^f(\tau) + 1$ is the sum of the volumes of these cubes. Let $V(\tau)$ denote the volume and $T(\tau)$ the area of the part of the ellipsoid situated in the positive octant $z_i \geq 0, i = 1, \dots, n-1$. Then

$$V(\tau) \leq A^f(\tau) + 1 \leq V(\tau) + (n-1)^{\frac{1}{2}} T(\tau),$$

where $(n-1)^{\frac{1}{2}}$ is the diagonal length of the unit cube (see, [5] or [26]). Since

$$V(\tau) = D_{n-1} 2^{-(n-1)} l_1 \dots l_{n-1} \left[\frac{h(\varrho^3 l_n^3 \tau^3)}{\pi l_n} \right]^{(n-1)},$$

by $h(t) \sim (\frac{t}{2})^{1/3}$ as $t \rightarrow +\infty$, we get that

$$V(\tau) \sim D_{n-1} (\sqrt[3]{16} \pi)^{-(n-1)} l_1 \dots l_{n-1} \varrho^{n-1} \tau^{n-1}, \quad \text{as } \tau \rightarrow +\infty.$$

Note that

$$T(\tau) \sim \text{constant} \cdot \tau^{n-2}.$$

It follows that

$$\lim_{\tau \rightarrow +\infty} \frac{A^f(\tau)}{\tau^{n-1}} = \omega_{n-1} (\sqrt[3]{16} \pi)^{-(n-1)} l_1 \dots l_{n-1} \varrho^{n-1},$$

i.e.,

$$(5.24) \quad A^f(\tau) \sim \frac{\omega_{n-1}}{(\sqrt[3]{16} \pi)^{(n-1)}} |\Gamma_\varrho^+| \varrho^{n-1} \tau^{n-1}, \quad \text{as } \tau \rightarrow +\infty,$$

where $|\Gamma_\varrho^+|$ denotes the area of the face Γ_ϱ^+ .

Next, we consider $A^0(\tau)$. Similarly,

$$(5.25) \quad \frac{2\eta^3 l_n^3}{\varrho^3 l_n^3} \left(\frac{(\sinh \eta l_n)(\cosh \eta l_n) + \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) \leq \tau^3,$$

is equivalent to

$$\eta l_n \leq h(\varrho^3 l_n^3 \tau^3),$$

i.e.,

$$\sum_{i=1}^{n-1} [m_i/l_i]^2 \leq \left(\frac{h(\varrho^3 l_n^3 \tau^3)}{\pi l_n} \right)^2, \quad m_i = 1, 2, 3, \dots$$

Similar to the argument for $A^f(\tau)$, we find (see also, §4 of [5]) that

$$\begin{aligned} \#\{(m_1, \dots, m_{n-1}) \mid \sum_{i=1}^{n-1} \left(\frac{m_i}{l_i}\right)^2 \leq \left(\frac{h(\varrho^3 l_n^3 \tau^3)}{\pi l_n}\right)^2, \ m_i = 1, 2, 3, \dots\} \\ \sim \frac{\omega_{n-1}}{(\sqrt[3]{16} \pi)^{(n-1)}} |\Gamma_\varrho^+| \varrho^{n-1} \tau^{n-1} \quad \text{as } \tau \rightarrow +\infty. \end{aligned}$$

i.e.,

$$(5.26) \quad \lim_{\tau \rightarrow +\infty} \frac{A^0(\tau)}{\tau^{n-1}} = \frac{\omega_{n-1}}{(\sqrt[3]{16} \pi)^{(n-1)}} |\Gamma_\varrho^+| \varrho^{n-1}.$$

Not that $\varrho = 0$ on $\partial D - \Gamma_\varrho^+$, by (5.7), (5.18), (5.24) and (5.26), we have

$$(5.27) \quad A(\tau) \sim \frac{\omega_{n-1} \tau^{n-1}}{(\sqrt[3]{16} \pi)^{(n-1)}} \int_{\Gamma_\varrho} \varrho^{n-1} ds \quad \text{as } \tau \rightarrow +\infty.$$

5.3. A cylinder D whose base is an n -polyhedron of \mathbb{R}^{n-1} having $n-1$ orthogonal plane surfaces and $g^{ik} = \delta^{ik}$.

Lemma 5.1. *Let $D^{(j)} = \Gamma_\varrho^{(j)} \times [0, l_n]$, $j = 1, 2$, where $\Gamma_\varrho^{(1)} = \{(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \mid x_i \geq 0 \text{ for } 1 \leq i \leq n-1, \text{ and } \sum_{i=1}^{n-1} \frac{x_i}{l_i} \leq 1\}$, and $\Gamma_\varrho^{(2)}$ is an $(n-1)$ -dimensional cube with side length $l = \max_{1 \leq i \leq n-1} l_i$. Assume that $\Gamma_0^{(j)} = \Gamma_\varrho^{(j)} \times \{l_n\}$, $j = 1, 2$. Assume also that ϱ is a positive constant on $\Gamma_\varrho^{(j)}$, $j = 1, 2$. If $l < l_n$, then*

$$(5.28) \quad \gamma_k^f(D^{(1)}) \geq \gamma_k^f(D^{(2)}) \quad \text{for } k = 1, 2, 3, \dots,$$

where $(\gamma_k^f(D^{(j)}))^3$ (similar to γ^3 of (3.23) in Theorem 3.8) is the k -th Steklov eigenvalue for the domain $D^{(j)}$.

Proof. Let $v_k^{(j)}$ be the k -th Neumann eigenfunction corresponding to $\alpha_k^{(j)}$ for the $(n-1)$ -dimensional domain $\Gamma_\varrho^{(j)}$, $(j = 1, 2)$, i.e.,

$$(5.29) \quad \begin{cases} \Delta v_k^{(j)} + \alpha_k^{(j)} v_k^{(j)} = 0 & \text{in } \Gamma_\varrho^{(j)}, \\ \frac{\partial v_k^{(j)}}{\partial \nu} = 0 & \text{on } \partial \Gamma_\varrho^{(j)}. \end{cases}$$

Put

$$u_k^{(j)}(x) = (v_k^{(j)}(x_1, \dots, x_{n-1}))(Z^{(j)}(x_n)) \quad \text{in } D^{(j)},$$

where $Z^{(j)}(x_n)$ is as in (4.13) with β being replaced by $\sqrt{\alpha_k^{(j)}}$. It is easy to verify that $u_k^{(j)}(x)$ satisfies

$$(5.30) \quad \begin{cases} \Delta^2 u_k^{(j)} = 0 & \text{in } D^{(j)}, \\ \frac{\partial u_k^{(j)}}{\partial \nu} = 0 & \text{on } \Gamma_\varrho^{(j)}, \\ u_k^{(j)} = \frac{\partial u_k^{(j)}}{\partial \nu} = 0 & \text{on } \Gamma_0^{(j)} \times \{l_n\}, \\ \frac{\partial u_k^{(j)}}{\partial \nu} = \frac{\partial(\Delta u_k^{(j)})}{\partial \nu} = 0 & \text{on } (\partial \Gamma_\varrho^{(j)}) \times [0, l_n], \\ \frac{\partial(\Delta u_k^{(j)})}{\partial \nu} - (\gamma_k^f(D^{(j)}))^3 \varrho^3 u_k^{(j)} = 0 & \text{on } \Gamma_\varrho^{(j)}. \end{cases}$$

with

$$(\gamma_k^f(D^{(j)}))^3 = \frac{2\sqrt{\alpha_k^{(j)}}}{\varrho^3} \left(\frac{(\sinh \sqrt{\alpha_k^{(j)}} l_n) \cosh \sqrt{\alpha_k^{(j)}} l_n + \sqrt{\alpha_k^{(j)}} l_n}{\sinh^2 \sqrt{\alpha_k^{(j)}} l_n - \alpha_k^{(j)} l_n^2} \right).$$

It follows from p. 437-438 of [5] that the k -th Neumann eigenvalue $\alpha_k^{(1)}$ for the domain $\Gamma_\varrho^{(1)}$ is at least as large as the k -th Neumann eigenvalue $\alpha_k^{(2)}$ for the domain $\Gamma_\varrho^{(2)}$. Recalling that $2s \left(\frac{(\sinh s) \cosh s + s}{\sinh^2 s - s^2} \right)$ is increasing when $s \geq 1$, we get

$$\left(\gamma_k^f(D^{(1)}) \right)^3 \geq \left(\gamma_k^f(D^{(2)}) \right)^3, \quad k = 1, 2, 3, \dots$$

if $l < l_n$. Here we have used the fact that $\sqrt{\alpha_k^{(2)}} l_n \geq 1$ since any Neumann eigenvalue for $\Gamma_\varrho^{(2)}$ has the form $\sum_{i=1}^{n-1} \left(\frac{m_i \pi}{l} \right)^2$. In other words, if $l < l_n$, then the number $A^f(\tau)$ of the eigenvalues less than or equal to a given bound τ^3 for the domain $D^{(1)}$ is at most equal to the corresponding number of the eigenvalues for the domain $D^{(2)}$. \square

Similarly, we can easily verify that the number $A^f(\tau)$ of the eigenvalues less than or equal to a given bound τ^3 for the n -dimensional rectangular parallelepiped D is never larger than the corresponding number for an n -dimensional rectangular parallelepiped of the same height whose base is an $(n-1)$ -dimensional cube and contains the base of D .

5.4. D is a cylinder and $g^{ik} = \delta^{ik}$.

Let D be an open n -dimensional cylinder in \mathbb{R}^n , whose boundary consists of an $(n-1)$ -dimensional cylindrical surface and two parallel plane surfaces perpendicular to the cylindrical surface. Assume that $g^{ik} = \delta^{ik}$ in the whole of \bar{D} , that Γ_ϱ includes at least one of the plane surfaces, which we call Γ_ϱ^+ , and that ϱ is positive constant on Γ_ϱ^+ and vanishes on $\Gamma_\varrho - \Gamma_\varrho^+$. We let the plane surface Γ_ϱ^+ be situated in the plane $x_n = 0$ and let another parallel surface Γ^{l_n} be situated in the plane $\{x \in \mathbb{R}^n | x_n = l_n\}$. We now divide the plane $x_n = 0$ into a net of $(n-1)$ -dimensional cubes, whose faces are parallel to the coordinate-planes in $x_n = 0$. Let $\Gamma_1, \dots, \Gamma_p$ be those open cubes in the net, closure of which are entirely contained in Γ_ϱ^+ , and let Q_{p+1}, \dots, Q_q be the remaining open cubes, whose closure intersect Γ_ϱ^+ . We may let the subdivision into cubes be so fine that, for every piece of the boundary of Γ_ϱ^+ which is contained in one of the closure cubes, the direction of the normal varies by less than a given angle ϑ , whose size will be determined later. (This can be accomplished by repeated halving of the side of cube.) We can make the side length l of each cube be less than l_n . Furthermore, let D_j , ($j = 1, \dots, p$), be the open n -dimensional rectangular parallelepiped with the cube Γ_j as a base and otherwise bounded by the "upper" plane surface Γ^{l_n} of the cylinder \bar{D} and planes parallel to the coordinate-planes $x_1 = 0, \dots, x_{n-1} = 0$ (cf. [26]).

We define the linear spaces of functions

$$\begin{aligned} K &= \{u | u \in Lip(\bar{D}) \cap H^2(D), \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_\varrho \cup \Gamma^{l_n}, u = 0 \text{ on } \Gamma - \Gamma_\varrho\}, \\ K_j^0 &= \{u_j | u_j \in Lip(\bar{D}_j) \cap H^2(D_j), \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_j \cup \Gamma_j^{l_n}, \text{ and } u = 0 \text{ on } \partial D_j - \Gamma_j\}, \\ &\quad (j = 1, \dots, p) \end{aligned}$$

with the inner products

$$\langle u, v \rangle = \int_D (\Delta u)(\Delta v) dR + \int_{\Gamma_e} \varrho^3 uv ds \quad \text{for } u, v \in K,$$

$$\langle u_j, v_j \rangle_j = \langle u_j, v_j \rangle_j^* + [u_j, v_j]_j = \int_{D_j} (\Delta u_j)(\Delta v_j) dR + \int_{\Gamma_j} \varrho^3 u_j v_j ds \quad \text{for } u_j, v_j \in K_j^0,$$

respectively. Closing K and K_j^0 with respect to the norms $\|u\| = \sqrt{\langle u, u \rangle}$ and $\|u_j\|_j = \sqrt{\langle u_j, u_j \rangle_j}$, we obtain the Hilbert spaces \mathcal{K} and \mathcal{K}_j^0 ($j = 1, \dots, p$), respectively. Clearly, the bilinear functional

$$\begin{aligned} [u, v] &= \int_{\Gamma_e} \varrho^3 uv ds \\ [u_j, v_j]_j &= \int_{\Gamma_j} \varrho^3 u_j v_j ds, \quad (j = 1, \dots, p), \end{aligned}$$

define self-adjoint, completely continuous transformations G and G_j^0 on \mathcal{K} and \mathcal{K}_j^0 by

$$(5.31) \quad \langle Gu, v \rangle = [u, v] \quad \text{for } u, v \text{ in } \mathcal{K},$$

$$(5.32) \quad \langle G_j^0 u_j, v_j \rangle_j = [u_j, v_j]_j \quad \text{for } u_j \text{ and } v_j \text{ in } \mathcal{K}_j^0,$$

respectively. By defining a space

$$\mathcal{K}^0 = \sum_{j=1}^p \oplus \mathcal{K}_j^0 = \{u^0 \mid u^0 = u_1 + \dots + u_p, \ u_j \in \mathcal{K}_j^0\}$$

with its inner product

$$(5.33) \quad \langle u^0, v^0 \rangle = \sum_{j=1}^p \langle u_j, v_j \rangle_j,$$

we find that the space \mathcal{K}^0 becomes a Hilbert space. If we define the transformation G^0 on \mathcal{K}^0 by

$$(5.34) \quad G^0 u^0 = G_1^0 u_1 + \dots + G_p^0 u_p \quad \text{for } u^0 = u_1 + \dots + u_p \text{ in } \mathcal{K}^0,$$

we see that G^0 is a self-adjoint, completely continuous transformation on \mathcal{K}^0 . If we put

$$(5.35) \quad [u^0, v^0] = \sum_{j=1}^p [u_j, v_j]_j,$$

we find by (5.32)—(5.35) that

$$(5.36) \quad \langle G^0 u^0, v^0 \rangle = [u^0, v^0] \quad \text{for all } u^0 \text{ and } v^0 \text{ in } \mathcal{K}^0.$$

We now define a mapping of \mathcal{K}^0 into \mathcal{K} . Let $u^0 = u_1 + \dots + u_p$, $u \in H_j^0$, be an element of \mathcal{K}^0 and define

$$(5.37) \quad u = \Pi^0 u^0,$$

where $u(x) = u_j(x)$, when $x \in \bar{D}_j$, and $u(x) = 0$, when $x \in \bar{D} - \cup_{j=1}^p \bar{D}_j$. Then $u \in \mathcal{K}$ and thus (5.37) defines a transformation Π^0 of $\mathcal{K}_1^0 \oplus \dots \oplus \mathcal{K}_p^0$ into \mathcal{K} . It is readily seen that

$$(5.38) \quad [\Pi^0 u^0, \Pi^0 v^0] = [u^0, v^0] \quad \text{for all } u^0 \text{ and } v^0 \text{ in } \mathcal{K}^0.$$

and

$$(5.39) \quad \langle G^0 u^0, v^0 \rangle = \langle G \Pi^0 u^0, \Pi^0 v^0 \rangle \quad \text{for all } u^0 \text{ and } v^0 \text{ in } \mathcal{K}^0.$$

From (5.38) and (5.39), we find by applying Corollary 1.4.1 of [26] that

$$\mu_k^0 \leq \mu_k \quad \text{for } k = 1, 2, 3, \dots$$

Therefore

$$(5.40) \quad A^0(\tau) \leq A(\tau).$$

The definition of G^0 implies that

$$(5.41) \quad G^0 \mathcal{K}_j^0 \subset \mathcal{K}_j^0, \quad (j = 1, \dots, p),$$

and

$$(5.42) \quad G^0 u^0 = G_j^0 u^0, \quad \text{when } u^0 \in \mathcal{K}_j^0.$$

From (5.35), (5.36), (5.41), (5.42) and Theorem 1.6 of [26], we obtain

$$(5.43) \quad A^0(\tau) = \sum_{j=1}^p A_j^0(\tau),$$

where $A_j^0(\tau)$ is the number of eigenvalues of the transformation G_j^0 on \mathcal{K}_j^0 which are greater or equal to $(1+\tau^3)^{-1}$. Because $\bar{D}_j, (j = 1, \dots, p)$, is an n -dimensional rectangular parallelepiped we find by (5.26) that

$$(5.44) \quad A_j^0(\tau) \sim \omega_{n-1} (\sqrt[3]{16} \pi)^{-(n-1)} |\Gamma_j| \varrho^{n-1} \tau^{n-1} \quad \text{as } \tau \rightarrow +\infty,$$

where $|\Gamma_j|$ denotes the area of the face Γ_j of D_j . By (5.43) and (5.44) we infer that

$$(5.45) \quad A^0(\tau) \sim \omega_{n-1} (\sqrt[3]{16} \pi)^{-(n-1)} \sum_{j=1}^p |\Gamma_j| \varrho^{n-1} \tau^{n-1} \quad \text{as } \tau \rightarrow +\infty.$$

Next, we shall calculate the upper estimate of $A(\tau)$. Let $\bar{P}_j, (j = p+1, \dots, q)$, be the n -dimensional rectangular parallelepiped with the cube \bar{Q}_j as a base and otherwise bounded by the “upper” plane surface Γ^{l_n} of the cylinder \bar{D} and planes parallel to the coordinate-planes $x_1 = 0, \dots, x_{n-1} = 0$. The intersection $\bar{P}_j \cap \bar{D}$ is a cylinder $\bar{D}_j, (j = p+1, \dots, q)$, with $\bar{\Gamma}_j := \bar{Q}_j \cap \bar{\Gamma}_\varrho$ as a basis. Then

$$(5.46) \quad \bar{D} = \sum_{j=1}^q \bar{D}_j.$$

We first define the linear spaces of functions

$$K^d = \{u \mid u \in Lip(\bar{D}) \cap H^2(D), \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_\varrho, u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma^{l_n}\},$$

$$K_j^d = \{u \mid u \in Lip(\bar{D}) \cap H^2(D_j), \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_j, u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_j^{l_n}\}, \quad (j = 1, \dots, q)$$

with the inner products

$$(5.47) \quad \langle u, v \rangle = \int_D (\Delta u)(\Delta v) dx + \int_{\Gamma_\varrho} \varrho^3 uv ds,$$

and

$$(5.48) \quad \langle u_j, v_j \rangle_j = \int_{D_j} (\Delta u_j)(\Delta v_j) dx + \int_{\Gamma_j} \varrho^3 u_j v_j ds,$$

respectively. Closing K^d and K_j^d with respect to the norms $\|u\| = \sqrt{\langle u, u \rangle}$ and $\|u_j\|_j = \sqrt{\langle u_j, u_j \rangle_j}$, we get Hilbert spaces \mathcal{K}^d and \mathcal{K}_j^d , ($j = 1, \dots, q$), and then we define the Hilbert space

$$(5.49) \quad \mathcal{K}^d = \sum_{j=1}^q \oplus \mathcal{K}_j^d = \{u^d \mid u^d = u_1 + \dots + u_q, \quad u_j \in \mathcal{K}_j^d\}$$

with its inner product

$$(5.50) \quad \langle u^d, v^d \rangle = \sum_{j=1}^q \langle u_j, v_j \rangle_j.$$

The bilinear functional

$$(5.51) \quad [u_j, v_j]_j = \int_{\Gamma_j} \varrho^3 u_j v_j ds, \quad (j = 1, \dots, q),$$

define a self-adjoint, completely continuous transformation G_j^d on \mathcal{K}_j^d given by

$$(5.52) \quad \langle G_j^d u_j, v_j \rangle_j = [u_j, v_j]_j \quad \text{for all } u_j \text{ and } v_j \text{ in } \mathcal{K}_j^d.$$

The self-adjoint, completely continuous transformation G^d on \mathcal{K}^d is defined by

$$(5.53) \quad G^d u^d = G_1^d u_1 + \dots + G_q^d u_q \quad \text{for } u^d = u_1 + \dots + u_q \text{ in } \mathcal{K}^d.$$

With

$$(5.54) \quad [u^d, v^d] = \sum_{j=1}^q [u_j, v_j]_j,$$

it follows from (5.50), (5.52)—(5.54) that

$$(5.55) \quad \langle G^d u^d, v^d \rangle = [u^d, v^d] \quad \text{for all } u^d \text{ and } v^d \text{ in } \mathcal{K}^d.$$

Now we define a mapping Π of \mathcal{K} into \mathcal{K}^d . Let $u \in K(D)$, and put

$$u^d = \Pi u = u_1 + \dots + u_q,$$

where $u_j(x) = u(x)$, when $x \in \bar{D}_j$. It can be easily verified that

$$(5.56) \quad \langle \Pi u, \Pi v \rangle = \langle u, v \rangle \quad \text{for all } u \text{ and } v \text{ in } \mathcal{K}.$$

and

$$(5.57) \quad \langle Gu, v \rangle = \langle G^d \Pi u, \Pi v \rangle \quad \text{for all } u \text{ and } v \text{ in } \mathcal{K}.$$

Combining (5.56), (5.57) and using Corollary 1.4.1 of [26], we obtain

$$\mu_k \leq \mu_k^d \quad \text{for } k = 1, 2, 3, \dots,$$

and hence

$$(5.58) \quad A(\tau) \leq A^d(\tau).$$

From $G^d \mathcal{K}_j^d \subset \mathcal{K}_j^d$, ($j = 1, \dots, q$), and $G^d u^d = G_j^d u^d$ when $u^d \in \mathcal{K}_j^d$, we get

$$(5.59) \quad A^d(\tau) = \sum_{j=1}^q A_j^d(\tau),$$

where $A_j^d(\tau)$ is the number of eigenvalues of the transformation G_j^d on \mathcal{K}_j^d which are greater than or equal to $(1 + \tau^3)^{-1}$. Also, we define $A_j^f(\tau)$ similar to (5.15) and (5.17), i.e.,

$$A_j^f(\tau) = \sum_{\mu_k^f \geq (1+\tau^3)^{-1}} 1 \quad \text{with} \quad \mu_k^f = \frac{1}{1 + \gamma_k^3},$$

where γ_k^3 is the k -th Steklov eigenvalue of the following problem

$$\begin{cases} \Delta^2 u_j = 0 & \text{in } D_j, \\ \frac{\partial u_j}{\partial \nu} = 0 & \text{on } \Gamma_j, \quad u_j = \frac{\partial u_j}{\partial \nu} = 0 & \text{on } \Gamma_j^{l_n}, \\ \frac{\partial u}{\partial \nu} = \frac{\partial(\Delta u_j)}{\partial \nu} = 0 & \text{on } \partial D_j - (\Gamma_j \cup \Gamma_j^{l_n}), \\ \frac{\partial(\Delta u_j)}{\partial \nu} - \gamma^3 \varrho^3 u_j = 0 & \text{on } \Gamma_j, \quad \varrho = \text{constant} > 0 & \text{on } \Gamma_\varrho^+. \end{cases}$$

From Theorem 3.8, it follows that

$$\gamma_k^3 \leq \kappa_k^3 \quad \text{for all } k \geq 1,$$

and hence

$$(5.60) \quad A_j^d(\tau) \leq A_j^f(\tau) \quad \text{for all } \tau \text{ and } j = 1, \dots, p,$$

where κ_k^3 is the k -th eigenvalue of the transformation G_j^d . Since $\bar{D}_j, (j = 1, \dots, p)$, is an n -dimensional rectangular parallelepiped, we find from (5.24) that

$$(5.61) \quad A_j^f(\tau) \sim \omega_{n-1} (\sqrt[3]{16} \pi)^{-(n-1)} |\Gamma_j| \varrho^{n-1} \tau^{n-1}, \quad (j = 1, \dots, p).$$

It remains to estimate $A_j^f(\tau), (j \geq p+1)$. According to the argument in p. 438-440 of [5], each of the $(n-1)$ -dimensional domains Γ_j is bounded either by $n-1$ orthogonal plane surfaces of the partition (the diameter of the intersection of any two plane surfaces lies between l and $3l$), and an $(n-2)$ -dimensional surface of the boundary $\partial\Gamma_\varrho$ (see, in two dimensional case, Figure 5 of p. 439 of [5]), or by $2n-3$ orthogonal plane surfaces of the partition (the diameter of the intersection of any two plane surfaces lies between l and $3l$), and a surface of the boundary $\partial\Gamma_\varrho$ (see, in two dimensional case, Figure 6 of p. 439 of [5]). The number $q-p$ is evidently smaller than a constant C/l^{n-2} , where C is independent of l and depends essentially on the area of the boundary $\partial\Gamma_\varrho$. Now, we take any point on the boundary surface of Γ_j and take the tangent plane through it. This tangent plane together with the plane parts of $\partial\Gamma_j$ bounds an n -polyhedron of \mathbb{R}^{n-1} with a vertex at which $n-1$ orthogonal plane surfaces meet (see, Figure 5 of p. 439 of [5] in two dimensions), e.g., if ϑ is sufficiently small it forms an $(n-1)$ -dimensional n -polyhedron of \mathbb{R}^n with a vertex having $n-1$ orthogonal plane surfaces (the diameter of the intersection of any two plane surfaces is also smaller than $4l$), or else an $(n-1)$ -dimensional $2(n-1)$ -polyhedron of \mathbb{R}^{n-1} (see, Figure 6 of p. 439 of [5] in two dimensional case), the diameter of the intersection of any two plane surfaces (except for the top inclined plane surface) of the $2(n-1)$ -polyhedron is also smaller than $4l$; The shape of the result domain depends on the type to which $\bar{\Gamma}_j$ belongs. We shall denote the result domains by S'_j . The domain Γ_j can always be deformed into the domain S'_j by a transformation of the form (2.5), as defined in Section 2. In the case of domains of the first type, let the intersection point of $n-1$ orthogonal plane surfaces be the pole of a system of pole coordinates $r, \theta_1, \theta_2, \dots, \theta_{n-2}$, and let $r = f(\theta_1, \theta_2, \dots, \theta_{n-2})$ be the equation of the boundary surface of

Γ_ϱ , $r = h(\theta_1, \theta_2, \dots, \theta_{n-2})$ the equation of the inclined plane surface of the n -polyhedron of \mathbb{R}^{n-1} having a vertex of $n-1$ orthogonal plane surfaces. Then the equations

$$\theta'_1 = \theta_1, \quad \theta'_2 = \theta_2, \quad \dots, \quad \theta'_{n-2} = \theta_{n-2}, \quad r' = r \frac{h(\theta_1, \theta_2, \dots, \theta_{n-2})}{f(\theta_1, \theta_2, \dots, \theta_{n-2})}$$

represents a transformation of the domain Γ_j into the n -polyhedron S'_j of \mathbb{R}^{n-1} . For a domain of the second type, let $x_{n-1} = h(x_1, \dots, x_{n-2})$ be the equation of top plane surface of the $2(n-1)$ -polyhedron and let $x_{n-1} = f(x_1, \dots, x_{n-2})$ be the equation of the boundary surface of Γ_ϱ . We then consider the transformation

$$x'_1 = x_1, \quad \dots, \quad x'_{n-2} = x_{n-2}, \quad x'_{n-1} = x_{n-1} \frac{h(x_1, \dots, x_{n-2})}{f(x_1, \dots, x_{n-2})}.$$

If we assume that the side length l of cube in the partition is sufficiently small, and therefore the rotation of the normal on the boundary surface is taken sufficiently small, then the transformations considered here evidently have precise the form (2.5), and the quantity denoted by ϵ in (2.5) is arbitrarily small. From Corollary to Theorem 10 of p. 423 of [5], we know that there exists a number $\delta > 0$ depending on ϵ and approaching zero with ϵ , such that

$$\left| \frac{\alpha_k(S'_j)}{\alpha_k(\Gamma_j)} - 1 \right| < \delta \quad \text{uniformly for all } k,$$

where $\alpha_k(\Gamma_j)$ and $\alpha_k(S'_j)$ are the k -th Neumann eigenvalues of Γ_j and S'_j , respectively. According to the argument as in the proof of Lemma 5.1, we see that

$$(\gamma_k^f(E_j))^3 = \frac{1}{\varrho^3 l_n^3} t(l_n \alpha_k(\Gamma_j)), \quad (\gamma_k^f(E'_j))^3 = \frac{1}{\varrho^3 l_n^3} t(l_n \alpha_k(S'_j)),$$

where $t(s)$ is given by (5.22), and $(\gamma_k^f(E_j))^3$ and $(\gamma_k^f(E'_j))^3$ (similar to γ^3 of (3.23)) are the k -th Steklov eigenvalue for the n -dimensional domains $E_j = \Gamma_j \times [0, l_n]$ and $E'_j = S'_j \times [0, l_n]$, respectively. Recalling that the function $t = t(s)$ is continuous and increasing for $s \geq 1$, and $\lim_{s \rightarrow +\infty} \frac{t(s)}{s} = \frac{1}{2}$, we get that there exists a constant $\delta' > 0$ depending on ϵ approaching zero with ϵ , such that

$$\left| \frac{(\gamma_k^f(E_j))^3}{(\gamma_k^f(E'_j))^3} - 1 \right| < \delta'.$$

In other words, the corresponding k -th eigenvalues for the n -dimensional domains $E_j = \Gamma_j \times [0, l_n]$ and $E'_j = S'_j \times [0, l_n]$ differ only by a factor which itself differs by a small amount from 1, uniformly for all k . Therefore, the same is true also for the corresponding numbers $A_{E_j}^f(\tau)$ and $A_{E'_j}^f(\tau)$ of the eigenvalues less or equal to the bound τ^3 .

The domain E'_j is either a cylinder whose base is an n -polyhedron of \mathbb{R}^{n-1} having $(n-1)$ orthogonal plane surfaces with its largest side length small than $4l$ or a cylinder whose base is a combination of such an n -polyhedron of \mathbb{R}^{n-1} and an $(n-1)$ -dimensional cube with side-length smaller than $3l$; it follows that if l is taken sufficiently small, the number $A_{E_j}^f(\tau)$ from some τ on satisfies the inequality

$$A_{E_j}^f(\tau) < C_1 l^{n-1} \tau^{n-1} + C_2 l^{n-2} \tau^{n-2}$$

where C_1, C_2 are constants, to be chosen suitably. Thus, $A_{E_j}^f(\tau)$ can be written as $A_{E_j}^f(\tau) = \theta(C_3 l^{n-1} \tau^{n-1} + C_4 l^{n-2} \tau^{n-2})$, where θ denotes a number between -1 and $+1$

and C_3, C_4 are constants independent of l, j and τ . It follows that

$$\sum_{j=p+1}^q A_{E_j}^f(\tau) = \tau^{n-1} [\theta C_3(q-p)l^{n-1} + \theta C_4(q-p)l^{n-2} \frac{1}{\tau}].$$

As pointed out before, $(q-p)l^{n-2} < C$; therefore, for sufficiently small l , $(q-p)l^{n-1}$ is arbitrarily small and we have the asymptotic relation

$$(5.62) \quad \lim_{\tau \rightarrow +\infty} \sum_{j=p+1}^q \frac{A_{E_j}^f(\tau)}{\tau^{n-1}} = \varsigma(l),$$

where $\varsigma(l) \rightarrow 0$ as $l \rightarrow 0$. For, we may choose the quantity l arbitrarily, and by taking a sufficiently small fixed l , make the factor of τ^{n-1} in the above equalities arbitrarily close to zero for sufficiently large τ . Since

$$(5.63) \quad A_{E_j}^d(\tau) \leq A_{E_j}^f(\tau) \quad \text{for } j = p+1, \dots, q,$$

we get

$$(5.64) \quad \lim_{\tau \rightarrow +\infty} \frac{\sum_{j=p+1}^q A_{E_j}^d(\tau)}{\tau^{n-1}} \leq \lim_{\tau \rightarrow +\infty} \frac{\sum_{j=p+1}^q A_{E_j}^f(\tau)}{\tau^{n-1}} = \varsigma(l).$$

Combining (5.40), (5.45), (5.58), (5.60), (5.61), (5.62), (5.63) and (5.64) we obtain

$$(5.65) \quad \begin{aligned} \omega_{n-1}(\sqrt[3]{16}\pi)^{-(n-1)} \varrho^{n-1} \sum_{j=1}^p |\Gamma_j| &\leq \liminf_{\tau \rightarrow \infty} \frac{A(\tau)}{\tau^{n-1}} \leq \overline{\lim}_{\tau \rightarrow \infty} \frac{A(\tau)}{\tau^{n-1}} \\ &\leq \left(\omega_{n-1}(\sqrt[3]{16}\pi)^{-(n-1)} \varrho^{n-1} \sum_{j=1}^p |\Gamma_j| \right) + \varsigma(l). \end{aligned}$$

Letting $l \rightarrow 0$, we immediately see that $\sum_{j=1}^p |\Gamma_j|$ tends to the area $|\Gamma_\varrho|$ of Γ_ϱ and $\lim_{l \rightarrow 0} \varsigma(l) = 0$. Therefore, (5.65) gives

$$(5.66) \quad A(\tau) \sim \frac{\omega_{n-1}}{(\sqrt[3]{16}\pi)^{(n-1)}} |\Gamma_\varrho| \varrho^{n-1} \tau^{n-1} \quad \text{as } \tau \rightarrow +\infty,$$

or

$$(5.67) \quad A(\tau) \sim \frac{\omega_{n-1} \tau^{n-1}}{(\sqrt[3]{16}\pi)^{(n-1)}} \int_{\Gamma_\varrho} \varrho^{n-1} ds \quad \text{as } \tau \rightarrow +\infty.$$

In the above argument, we first made the assumption that the boundary $\partial\Gamma_\varrho$ of Γ_ϱ is smooth. However, the corresponding discussion and result remain essentially valid if $\partial\Gamma_\varrho$ is composed of a finite number of $(n-2)$ dimensional smooth surfaces.

6. PROOFS OF MAIN RESULTS

Lemma 6.1. *Let g^{il} and g'^{il} be two metric tensors on manifold \mathcal{M} such that*

$$(6.1) \quad |g^{il} - g'^{il}| < \epsilon, \quad i, l = 1, \dots, n$$

and

$$(6.2) \quad \left| \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} (\sqrt{|g|} g^{il}) - \frac{1}{\sqrt{|g'|}} \frac{\partial}{\partial x_i} (\sqrt{|g'|} g'^{il}) \right| \leq \epsilon, \quad i, l = 1, \dots, n$$

for all points in \bar{D} , where D is a bounded domain in \mathcal{M} (see, Section 3). Let

$$\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq \dots > 0 \quad \text{and} \quad \mu'_1 \geq \mu'_2 \geq \dots \geq \mu'_n \geq \dots > 0$$

be positive eigenvalues of G and G' , respectively, where G and G' are given by

$$\begin{aligned} \langle Gu, v \rangle &= \int_{\Gamma_e} \varrho^3 uv \, ds, \quad \text{for } u \text{ and } v \text{ in } \mathcal{K}, \\ \langle G'u, v \rangle' &= \int_{\Gamma_e} \varrho^3 uv \, ds', \quad \text{for } u \text{ and } v \text{ in } \mathcal{K}'. \end{aligned}$$

Then, for $k = 1, 2, 3, \dots$,

$$(6.3) \quad \begin{aligned} & \frac{(1 + \epsilon \tilde{M})^{-n - \frac{1}{2}}}{(\max\{(1 + \epsilon M), (1 + \epsilon \tilde{M})^{1/2}\})^{\mu_k}} \mu_k \leq \mu'_k \\ & \leq \frac{(1 + \epsilon \tilde{M})^{n + \frac{1}{2}}}{(\min\{(1 - \epsilon M), (1 + \epsilon \tilde{M})^{-1/2}\})^{\mu_k}}, \end{aligned}$$

where \tilde{M} and M are constants depending only on $g, g', \partial g, \partial g'$ and \bar{D} .

Proof. It follows from (6.1) that there exists a positive constant \tilde{M} independent of ϵ and depending only on g^{ij}, g'^{ij} and \bar{D} such that

$$(1 + \epsilon \tilde{M})^{-1} \sum_{i,l=1}^n g^{il} t_i t_l \leq \sum_{i,l=1}^n g'^{il} t_i t_l \leq (1 + \epsilon \tilde{M}) \sum_{i,l=1}^n g^{il} t_i t_l$$

for all points in \bar{D} and all real numbers t_1, \dots, t_n . Thus we have

$$(1 + \epsilon \tilde{M})^{-n/2} \sqrt{|g|} \leq \sqrt{|g'|} \leq (1 + \epsilon \tilde{M})^{n/2} \sqrt{|g|},$$

which implies (see, p. 64-65 of [26]) that

$$(6.4) \quad (1 + \epsilon \tilde{M})^{-n/2} dR \leq dR' \leq (1 + \epsilon \tilde{M})^{n/2} dR$$

and

$$(6.5) \quad (1 + \epsilon \tilde{M})^{-(n+1)/2} ds \leq ds' \leq (1 + \epsilon \tilde{M})^{(n+1)/2} ds.$$

Thus

$$(6.6) \quad (1 + \epsilon \tilde{M})^{-(n+1)/2} [u, u] \leq [u, u]' \leq (1 + \epsilon \tilde{M})^{(n+1)/2} [u, u].$$

Putting

$$\omega_{il} = g'^{il} - g^{il}, \quad \theta_{il} = \frac{1}{\sqrt{|g'|}} \frac{\partial}{\partial x_i} (\sqrt{|g'|} g'^{il}) - \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} (\sqrt{|g|} g^{il}),$$

we immediately see that

$$\max_{x \in \bar{D}} |\omega_{il}| \leq \epsilon \quad \text{and} \quad \max_{x \in \bar{D}} |\theta_{il}| \leq \epsilon.$$

Thus, for any $u \in \mathcal{K}(D)$ or $u \in \mathcal{K}^d(D)$, we have

$$\Delta_{g'}u = \sum_{i,l=1}^n (\omega_{il} + g^{il}) \frac{\partial^2 u}{\partial x_i \partial x_l} + \sum_{i,l=1}^n \left[\theta_{il} + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} (\sqrt{|g|} g^{il}) \right] \frac{\partial u}{\partial x_l},$$

so that

$$\Delta_{g'}u - \Delta_g u = \sum_{i,l=1}^n \left[\omega_{il} \frac{\partial^2 u}{\partial x_i \partial x_l} + \theta_{il} \frac{\partial u}{\partial x_l} \right]$$

It follows that

$$|\Delta_{g'}u - \Delta_g u| \leq \epsilon (M_1 |\nabla_g^2 u| + M_2 |\nabla_g u|),$$

where $|\nabla_g^2 u|^2$ is defined in an invariant ways as

$$|\nabla_g^2 u|^2 = \nabla^l \nabla^k u \nabla_l \nabla_k u = g^{pl} g^{kq} \left(\frac{\partial^2 u}{\partial x^k \partial x^l} - \Gamma_{kl}^m \frac{\partial u}{\partial x^m} \right) \left(\frac{\partial^2 u}{\partial x^p \partial x^q} - \Gamma_{pq}^r \frac{\partial u}{\partial x^r} \right),$$

and M_1 and M_2 are constants depending only on $g, g', \partial g, \partial g'$ and \bar{D} . Thus,

$$(6.7) \quad \int_D |\Delta_{g'}u - \Delta_g u|^2 dR \leq 2\epsilon^2 \left(M_1^2 \int_D |\nabla_g^2 u|^2 dR + M_2^2 \int_D |\nabla_g u|^2 dR \right).$$

Set

$$(6.8) \quad \tilde{\Lambda}_1^0(D) = \inf_{v \in K(D), \int_D |\nabla_g v|^2 dR = 1} \frac{\int_D |\Delta_g v|^2 dR}{\int_D |\nabla_g v|^2 dR},$$

$$(6.9) \quad \tilde{\Lambda}_1^d(D) = \inf_{v \in K^d(D), \int_D |\nabla_g v|^2 dR = 1} \frac{\int_D |\Delta_g v|^2 dR}{\int_D |\nabla_g v|^2 dR},$$

and the spaces $K(D)$ and $K^d(D)$ are as in Section 3. Furthermore, set

$$(6.10) \quad \Theta_1^0(D) = \inf_{v \in K(D), \int_D |\nabla_g^2 v|^2 dR = 1} \frac{\int_D |\Delta_g v|^2 dR}{\int_D |\nabla_g^2 v|^2 dR},$$

$$(6.11) \quad \Theta_1^d(D) = \inf_{v \in K^d(D), \int_D |\nabla_g^2 v|^2 dR = 1} \frac{\int_D |\Delta_g v|^2 dR}{\int_D |\nabla_g^2 v|^2 dR}.$$

Clearly, $\tilde{\Lambda}_1^0(D) \geq \tilde{\Lambda}_1^d(D)$, and $\Theta_1^0(D) \geq \Theta_1^d(D)$. Similar to the proofs of Lemmas 2.1, 2.2, it is easy to prove that the existence of the minimizers to (6.9) and (6.11), respectively. Therefore, we have that $\tilde{\Lambda}_1^d(D) > 0$ and $\Theta_1^d(D) > 0$ (Suppose by contradiction that $\tilde{\Lambda}_1^d(D) = 0$ and $\Theta_1^d(D) = 0$. Then $\Delta_g u = 0$ in D for the corresponding minimizer $u \in K^d(D)$ in two cases. By applying Holmgren's uniqueness theorem for the minimizer $u \in K^d(D)$ in each case, we immediately see that $u \equiv 0$ in D . This contradicts the assumption $\int_D |\nabla_g u|^2 dR = 1$ or $\int_D |\nabla_g^2 u|^2 dR = 1$ for the minimizer $u \in K^d(D)$ in the corresponding cases). Combining these inequalities, we obtain

$$\int_D |\Delta_{g'}u - \Delta_g u|^2 dR \leq 2\epsilon^2 \left(\frac{M_1^2}{\Theta_1^0(D)} + \frac{M_2^2}{\tilde{\Lambda}_1^0(D)} \right) \int_D |\Delta_g u|^2, \quad \text{for } u \in K(D)$$

and

$$\int_D |\Delta_{g'}u - \Delta_g u|^2 dR \leq 2\epsilon^2 \left(\frac{M_1^2}{\Theta_1^d(D)} + \frac{M_2^2}{\tilde{\Lambda}_1^d(D)} \right) \int_D |\Delta_g u|^2 \quad \text{for } u \in K^d(D).$$

Thus we have that, for all $u \in K(D)$ or $u \in K^d(D)$,

$$(1 - \epsilon M) \int_D |\Delta_g u|^2 dR \leq \int_D |\Delta_{g'} u|^2 dR \leq (1 + \epsilon M) \int_D |\Delta_g u|^2 dR,$$

where M is a constant depending only $g, g', \partial g, \partial g'$ and \bar{D} . Combining this and (6.4) we get

$$(1 - \epsilon M)(1 + \epsilon \tilde{M})^{-\frac{n}{2}} \int_D |\Delta_g u|^2 dR \leq \int_D |\Delta_{g'} u|^2 dR' \leq (1 + \epsilon M)(1 + \epsilon \tilde{M})^{\frac{n}{2}} \int_D |\Delta_g u|^2 dR.$$

That is,

$$(6.12) \quad (1 - \epsilon M)(1 + \epsilon \tilde{M})^{-\frac{n}{2}} \langle u, u \rangle^* \leq \langle u, u \rangle'^* \leq (1 + \epsilon M)(1 + \epsilon \tilde{M})^{\frac{n}{2}} \langle u, u \rangle^*.$$

By (6.6) and (6.12) we obtain that, for all $u \in K(D)$ or $u \in K^d(D)$,

$$\begin{aligned} & \frac{(1 + \epsilon \tilde{M})^{-(n+1)/2} [u, u]}{(\max\{(1 + \epsilon M)(1 + \epsilon \tilde{M})^{\frac{n}{2}}, (1 + \epsilon \tilde{M})^{(n+1)/2}\}) (\langle u, u \rangle^* + [u, u])} \leq \frac{[u, u]'}{\langle u, u \rangle'^* + [u, u]'} \\ & \leq \frac{(1 + \epsilon \tilde{M})^{(n+1)/2} [u, u]}{(\min\{(1 - \epsilon M)(1 + \epsilon \tilde{M})^{-\frac{n}{2}}, (1 + \epsilon \tilde{M})^{-(n+1)/2}\}) (\langle u, u \rangle^* + [u, u])}, \end{aligned}$$

which implies (6.3). \square

Remark 6.2. Let $\tilde{\Gamma}$ and Γ be two bounded domains in \mathbb{R}^{n-1} , and let $\tilde{\Gamma}$ is similar to Γ (in the elementary sense of the term; the length of any line in $\tilde{\Gamma}$ is to the corresponding length in Γ as h to 1), and let $\Gamma_0 = \Gamma \times \{\sigma\}$ and $\tilde{\Gamma}_0 = \tilde{\Gamma} \times \{h\sigma\}$. It is easy to verify that

$$\tilde{\Lambda}_1^d(\tilde{D}) = h^{-2} \tilde{\Lambda}_1^d(D), \quad \Theta_1^d(\tilde{D}) = \Theta_1^d(D),$$

where $D = \Gamma \times [0, \sigma]$, $\tilde{D} = \tilde{\Gamma} \times [0, h\sigma]$, and $\tilde{\Lambda}_1^d(D)$ and $\Theta_1^d(D)$ are defined as in (6.9) and (6.11), respectively.

Lemma 6.3. Let G and G' be the continuous linear transformations defined by

$$\langle Gu, v \rangle = \int_{\Gamma_e} \varrho^3 uv ds \quad \text{for } u \text{ and } v \text{ in } K(D) \text{ or } K^d(D)$$

and

$$\langle G'u, v \rangle' = \int_{\Gamma_e} \varrho'^3 uv ds \quad \text{for } u \text{ and } v \text{ in } K(D) \text{ or } K^d(D),$$

respectively. Let

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq \cdots > 0 \quad \text{and} \quad \mu'_1 \geq \mu'_2 \geq \cdots \geq \mu'_k \geq \cdots > 0$$

be the positive eigenvalues of G and G' , respectively. If $\varrho \leq \varrho'$, then

$$(6.13) \quad \mu_k \leq \mu'_k \quad \text{for } k = 1, 2, 3, \dots$$

Proof. Since $\varrho \leq \varrho'$, we see that for any $u \in K(D)$ or $K^d(D)$,

$$\frac{\langle Gu, u \rangle}{\langle u, u \rangle^* + \int_{\Gamma_e} \varrho^3 u^2 ds} \leq \frac{\int_{\Gamma_e} \varrho^3 u^2 ds}{\langle u, u \rangle^* + \int_{\Gamma_e} \varrho'^3 u^2 ds} = \frac{\langle G'u, u \rangle'}{\langle u, u \rangle'^* + [u, u]'},$$

which implies (6.13). \square

Proof of Theorem 1.1. a) First, let (\mathcal{M}, g) be a real analytic Riemannian manifold, and let the boundary ∂D of D be $C^{2,\varepsilon}$ -smooth. As in [26], we divide the domain \bar{D} into subdomains in the following manner. It is clear that ∂D is the union of the portions $\bar{\Gamma}_1, \dots, \bar{\Gamma}_p$ (without common inner point on the surface). Let U be a coordinate neighborhood which contains $\bar{\Gamma}_j$, let $x_i = x_i(Q)$ and $a_i = a_i(\nu_Q)$ be the coordinates of a point Q in $\bar{\Gamma}_j$ and the interior Riemannian normal ν_Q at Q , respectively. We define the subdomain D_j and surface Γ_j^σ by

$$D_j = \{P | x(P) = x(Q) + \xi_n a(\nu_Q), \quad Q \in \Gamma_j, \quad 0 < \xi_n < \sigma\}$$

and

$$\Gamma_j^\sigma = \{P | x(P) = x(Q) + \sigma a(\nu_Q), \quad Q \in \Gamma_j\},$$

where σ is a positive constant. The closure of D_j is

$$(6.14) \quad \bar{D}_j = \{P | x(P) = x(Q) + \xi_n a(\nu_Q), \quad Q \in \bar{\Gamma}_j, \quad 0 \leq \xi_n \leq \sigma\}.$$

By the assumption, each $\bar{\Gamma}_j$, which is contained in a coordinate neighborhood, can be represented by equations

$$(6.15) \quad x_i = \psi_i(\xi_1, \dots, \xi_{n-1})$$

with $C^{2,\varepsilon}$ -smooth functions ψ_i , i.e., it is the image of the closure $\bar{\Upsilon}_j$ of an open domain Υ_j of \mathbb{R}^{n-1} . Hence, if σ is sufficiently small, the definitions have a sense and the formula

$$(6.16) \quad x(P) = x(Q) + \xi_n a(\nu_Q), \quad Q \in \bar{\Gamma}_j, \quad 0 \leq \xi_n \leq \sigma$$

defines a $C^{2,\varepsilon}$ -smooth homeomorphism of a neighborhood of the image of \bar{D}_j in \mathbb{R}^n given by the coordinates x and a neighborhood U_j of the closed cylinder \bar{F}_j in \mathbb{R}^n defined by $\bar{F}_j = \{\xi | (\xi_1, \dots, \xi_{n-1}) \in \bar{\Upsilon}_j, \quad 0 \leq \xi_n \leq \sigma\}$. (Therefore all Lemmas of Section 2 are true for every \bar{D}_j). Moreover, the domains $\bar{D}_1, \dots, \bar{D}_p$ have no common inner points and the remainder $D_0 = D - \cup_{j=1}^p \bar{D}_j$ of D has a finite number of connected parts. Note that the boundary of \bar{D}_0 contains no part of Γ_ϱ with measure > 0 .

Define the space $N = N(D)$, \mathcal{N} and the transformation G on \mathcal{N} as in Section 3. We shall investigate the asymptotic behavior of $A(\tau)$ with regard to transformation G on space \mathcal{N} . Moreover, we define the function spaces

$$\begin{aligned} K_j^0 &= \{u_j | u_j \in Lip(\bar{D}_j) \cap H^2(D_j), \quad \frac{\partial u_j}{\partial \nu} = 0 \text{ on } \Gamma_j \cup \Gamma_j^\sigma, \text{ and } u = 0 \text{ on } \partial D_j - \Gamma_j\}, \\ H_0^0 &= \{u_0 | Lip(\bar{D}_0) \cap H^2(D_0), \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D_0\}, \\ K_j^d &= \{u_j | u_j \in Lip(\bar{D}_j) \cap H^2(D_j), \quad \frac{\partial u_j}{\partial \nu} = 0 \text{ on } \Gamma_j, \quad u_j = \frac{\partial u_j}{\partial \nu} = 0 \text{ on } \Gamma_j^\sigma\}, \\ &\quad (j = 0, 1, \dots, p), \end{aligned}$$

and the bilinear functionals

$$(6.17) \quad \langle u_j, u_j \rangle_j^* = \int_{D_j} |\Delta_g u_j|^2 dR, \quad (j = 0, 1, \dots, p),$$

$$(6.18) \quad [u_j, v_j]_j = \int_{\Gamma_j} \varrho^3 u_j v_j ds, \quad (j = 1, \dots, p), \quad [u_0, v_0] = 0,$$

and

$$(6.19) \quad \langle u_j, v_j \rangle_j = \langle u_j, v_j \rangle_j^* + [u_j, v_j]_j, \quad (j = 0, 1, \dots, p),$$

where $u_j, v_j \in K_j^0$ or K_j^d . Closing K_j^0 and K_j^d with respect to the norm $|u_j|_j = \sqrt{\langle u_j, u_j \rangle_j}$, we get the Hilbert spaces \mathcal{K}_j^0 and \mathcal{K}_j^d , ($j = 0, 1, \dots, p$). Then, in the same manner as in Section 5 we can define the Hilbert \mathcal{K}^0 and \mathcal{K}^d , and define the positive, completely continuous transformations G^0 , G^d , G_j^0 and G_j^d on \mathcal{K}^0 , \mathcal{K}^d , \mathcal{K}_j^0 and \mathcal{K}_j^d , respectively. Furthermore, we can prove

$$(6.20) \quad A^0(\tau) \leq A(\tau) \leq A^d(\tau) \quad \text{for all } \tau,$$

and

$$(6.21) \quad A^0(\tau) = \sum_{j=0}^p A_j^0(\tau), \quad A^d(\tau) = \sum_{j=0}^p A_j^d(\tau),$$

where $A^0(\tau)$, $A^d(\tau)$, $A_j^0(\tau)$ and $A_j^d(\tau)$ are the numbers of eigenvalues of the transformations G^0 , G^d , G_j^0 and G_j^d on \mathcal{K}^0 , \mathcal{K}^d , \mathcal{K}_j^0 and \mathcal{K}_j^d which are greater than or equal to $(1 + \tau^3)^{-1}$, respectively.

Since $[u_0, u_0]_0 = 0$ for all $u_0, v_0 \in K_0^0$ or K_0^d and $\langle G_0^0 u_0, u_0 \rangle_0 = \langle G_0^d u_0, u_0 \rangle_0 = [u_0, u_0]_0$, we find fairly easily that $G_0^0 = G_0^d = 0$, so that $A_0^0(\tau) = A_0^d(\tau) = 0$, ($\tau \geq 0$). Thus, we have to estimate $A_j^0(\tau)$ and $A_j^d(\tau)$ for those domains D_j , where $\int_{\Gamma_j} \varrho^3 ds > 0$.

We can choose a finer subdivision of ∂D by subdividing the domains $\tilde{\Upsilon}_j$ into smaller ones, e.g. by means of a cubical net in the coordinates ξ . According to p. 71 of [26], by a linear transformation of the coordinates we can choose a new coordinate system η such that

$$g^{il}(\bar{\eta}) = \delta^{il}, \quad (i, l = 1, \dots, n),$$

for one point $\bar{\eta}$ in the mapping T_j of Υ_j , and

$$(6.22) \quad x_k(P) = \psi_k(\eta_1, \dots, \eta_{n-1}) + \eta_n a(\nu(\eta_1, \dots, \eta_{n-1})),$$

for $(\eta_1, \dots, \eta_{n-1}) \in \bar{T}_j$, $0 \leq \eta_n \leq \sigma$

defines a $C^{2,\varepsilon}$ -smooth homeomorphism from \bar{E}_j to \bar{D}_j , where $\bar{E}_j = \{\eta = (\eta_1, \dots, \eta_n) | (\eta_1, \dots, \eta_{n-1}) \in \bar{T}_j, 0 \leq \eta_n \leq \sigma\}$ is a cylinder in \mathbb{R}^n . (This can also be realized by choosing a (Riemannian) normal coordinates system at the point $\bar{\eta} \in T_j$ for the manifold (\mathcal{M}, g) (see, for example, p. 77 of [14]) such that $a(\nu(\eta)) = (0, \dots, 0, 1)$ and by using the mapping (6.22)). If we denote the new subdomains of ∂D by $\bar{\Gamma}_j$ as before, it is clear that we can always choose them and σ (see p. 71 of [26]), so that,

$$(6.23) \quad |g'^{il}(\eta') - g^{il}(\bar{\eta})| < \epsilon, \quad i, l = 1, \dots, n,$$

$$(6.24) \quad \left| \frac{1}{\sqrt{|g(\eta')|}} \frac{\partial}{\partial x_i} (\sqrt{|g(\eta')|} g^{il}(\eta')) - \frac{1}{\sqrt{|g(\bar{\eta})|}} \frac{\partial}{\partial x_i} (\sqrt{|g(\bar{\eta})|} g^{il}(\bar{\eta})) \right| < \epsilon,$$

$i, l = 1, \dots, n,$

for any given $\epsilon > 0$, and all points $\eta' \in \bar{E}_j$. The inequalities (6.23) imply that

$$(6.25) \quad (1 + \tilde{M}_j \epsilon)^{-1} \sum_{i=1}^n t_i^2 \leq \sum_{i,l=1}^n g^{il}(\eta') t_i t_l \leq (1 + \tilde{M}_j \epsilon) \sum_{i=1}^n t_i^2$$

for all points $\eta' \in \bar{E}_j$ and all real numbers t_1, \dots, t_n , where \tilde{M}_j is a positive constant depending only on g^{il} and \bar{E}_j (cf. Lemma 6.1). This and formula (128) of [26] say that

$$(6.26) \quad (1 + \epsilon \tilde{M}_j)^{-n/2} |T_j| \leq |\Gamma_j| \leq (1 + \epsilon \tilde{M}_j)^{n/2} |T_j|,$$

where

$$|\Gamma_j| = \int_{T_j} \sqrt{g(\eta)} d\eta_1 \cdots d\eta_{n-1}, \quad |T_j| = \int_{T_j} d\eta \cdots d\eta_{n-1}$$

are the Riemannian and Euclidean areas of Γ_j and $|T_j|$, respectively,

Now, we consider the Hilbert spaces \mathcal{K}_j^0 and \mathcal{K}_j^d . When transported to \bar{E}_j , the underlying incomplete function spaces K_j^0 and K_j^d are

$$K_j^0 = \{u | u \in Lip(\bar{E}_j) \cap H^2(E_j), \frac{\partial u_j}{\partial \nu} = 0 \text{ on } T_j \cup T_j^\sigma, \text{ and } u = 0 \text{ on } \partial E_j - T_j\}$$

and

$$K_j^d = \{u_j | u_j \in Lip(\bar{E}_j) \cap H^2(E_j), \frac{\partial u_j}{\partial \nu} = 0 \text{ on } T_j, u_j = \frac{\partial u_j}{\partial \nu} = 0 \text{ on } T_j^\sigma\},$$

respectively. The inner product, which is similar to Section 5, is defined by

$$\langle u, v \rangle_j = \int_{E_j} (\Delta_g u)(\Delta_g v) \sqrt{g(\eta)} d\eta_1 \cdots d\eta_n + \int_{T_j} \varrho^3 uv \sqrt{g(\eta)} d\eta_1 \cdots d\eta_{n-1}$$

and the transformations G_j^0 and G_j^d are defined by

$$\langle G_j^0 u, v \rangle_j = \int_{T_j} \varrho^3 uv \sqrt{g(\eta)} d\eta_1 \cdots d\eta_{n-1}, \quad \text{for } u, v \text{ in } \mathcal{K}_j^0,$$

and

$$\langle G_j^d u, v \rangle_j = \int_{T_j} \varrho^3 uv \sqrt{g(\eta)} d\eta_1 \cdots d\eta_{n-1}, \quad \text{for } u, v \text{ in } \mathcal{K}_j^d,$$

respectively.

Let

$$(6.27) \quad \underline{\varrho}_j = \inf_{\bar{\Gamma}_j} \varrho \quad \text{and} \quad \bar{\varrho}_j = \sup_{\bar{\Gamma}_j} \varrho,$$

and let us introduce the inner products

$$\langle \underline{u}, \underline{v} \rangle_j = \int_{E_j} (\Delta u)(\Delta v) d\eta_1 \cdots d\eta_n + \int_{T_j} \underline{\varrho}_j^3 uv d\eta_1 \cdots d\eta_{n-1}$$

and

$$\overline{\langle u, v \rangle}_j = \int_{E_j} (\Delta u)(\Delta v) d\eta_1 \cdots d\eta_n + \int_{T_j} \bar{\varrho}_j^3 uv d\eta_1 \cdots d\eta_{n-1}$$

in the spaces K_j^0 and K_j^d , respectively. By closing these spaces in the corresponding norms, we get Hilbert spaces $\underline{\mathcal{K}}_j^0$ and $\bar{\mathcal{K}}_j^d$. Furthermore, we obtain the positive, completely continuous transformations \underline{G}_j^0 and \bar{G}_j^d on $\underline{\mathcal{K}}_j^0$ and $\bar{\mathcal{K}}_j^d$, which are given by

$$(6.28) \quad \underline{\langle G_j^0 u, v \rangle}_j = \int_{T_j} \underline{\varrho}_j^3 uv d\eta_1 \cdots d\eta_{n-1}, \quad \text{for } u \text{ and } v \text{ in } \underline{\mathcal{K}}_j^0$$

and

$$(6.29) \quad \overline{\langle G_j^d u, v \rangle}_j = \int_{T_j} \bar{\varrho}_j^3 uv d\eta_1 \cdots d\eta_{n-1}, \quad \text{for } u \text{ and } v \text{ in } \bar{\mathcal{K}}_j^d,$$

respectively.

Let $\mu_k(G_j^0)$ be the k -th positive eigenvalue of G_j^0 and so on. According to Lemma 6.1 and Remark 6.2, $\Lambda_1^d(D_j)$ and $\Theta_1^d(D_j)$ have uniformly positive lower bound when repeated taking finer division of D (In fact, by repeated halving the side length of every rectangular parallelepiped in the partition net of the coordinates η for each cylinder E_j , we see that $\bar{\Lambda}_1^d(D_j)$ will tend to $+\infty$, and that $\Theta_1^d(D_j)$ will have a positive lower bound). This implies that the corresponding positive constants \tilde{M}_j and M_j have uniformly upper bound when we further divide the domain D into finer a division, where \tilde{M}_j is defined as before, and M_j is a constant independent of ϵ and depending only on g , ∂g and \bar{E}_j as in Lemma 6.1. Denote by $c_j(\epsilon)$ the maximum value of $(1 + \epsilon\tilde{M}_j)^{n+\frac{1}{2}} (\min\{(1 - \epsilon M), (1 + \epsilon\tilde{M})^{-\frac{1}{2}}\})^{-1}$ and $(1 + \epsilon\tilde{M}_j)^{n+\frac{1}{2}} (\max\{(1 + \epsilon M), (1 + \epsilon\tilde{M})^{\frac{1}{2}}\})$. Obviously, $c_j(\epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0$. By virtue of (6.23) and (6.2), it follows from Lemmas 6.1 and 6.3 that

$$(6.30) \quad \mu_k(G_j^d) \leq c_j(\epsilon) \mu_k(\bar{G}_j^d)$$

and

$$(6.31) \quad \mu_k(G_j^0) \geq c_j(\epsilon)^{-1} \mu_k(\underline{G}_j^0),$$

so that

$$A_j^d(\tau) \leq \bar{A}_j^d(c_j(\epsilon)\tau + c_j(\epsilon) - 1)$$

and

$$(6.32) \quad A_j^0(\tau) \geq \underline{A}_j^0(c_j(\epsilon)^{-1}\tau + c_j(\epsilon)^{-1} - 1)$$

where $\bar{A}_j^d(\tau)$ and $\underline{A}_j^0(\tau)$ are the numbers of eigenvalues of the transformation \bar{G}_j^d and \underline{G}_j^0 which are greater than or equal to $(1 + \tau^3)^{-1}$, respectively. By (6.20) and (6.21), we obtain

$$(6.33) \quad \begin{aligned} \sum_j \underline{A}_j^0(c_j(\epsilon)^{-1}\tau + c_j(\epsilon)^{-1} - 1) &\leq A(\tau) \\ &\leq \sum_j \bar{A}_j^d(c_j(\epsilon)\tau + c_j(\epsilon) - 1). \end{aligned}$$

Finally, we shall apply the results of Section 5 to estimate $\underline{A}_j^0(\tau)$ and $\bar{A}_j^d(\tau)$. Note that

$$(6.34) \quad \bar{A}_j^d(\tau) \leq \bar{A}_j^f(\tau) \quad \text{for all } \tau > 0,$$

where \bar{A}_j^f is defined similar to (5.15)–(5.17). Formula (5.44), (5.61) and (6.34) imply that

$$(6.35) \quad \lim_{\tau \rightarrow +\infty} \frac{\underline{A}_j^0(\tau)}{\tau^{n-1}} \geq \omega_{n-1} (\sqrt[3]{16} \pi)^{-(n-1)} |T_j| \underline{\varrho}_j^{n-1}$$

and

$$(6.36) \quad \lim_{\tau \rightarrow +\infty} \frac{\bar{A}_j^f(\tau)}{\tau^{n-1}} \leq \omega_{n-1} (\sqrt[3]{16} \pi)^{-(n-1)} |T_j| \bar{\varrho}_j^{n-1},$$

where $|T_j|$ is the Euclidean area of T_j . Combining (6.33), (6.34), (6.35), (6.36), (6.20), (6.21), (5.64) and (5.65) we find

$$\lim_{\tau \rightarrow \infty} A(\tau) \tau^{-(n-1)} \leq \omega_{n-1} (\sqrt[3]{16} \pi)^{-(n-1)} \tilde{c}_j(\epsilon) \sum_j \bar{\varrho}_j^{n-1} |\Gamma_j|,$$

and

$$\lim_{\tau \rightarrow \infty} A(\tau) \tau^{-(n-1)} \geq \omega_{n-1} (\sqrt[3]{16} \pi)^{-(n-1)} \tilde{c}_j(\epsilon)^{-1} \sum_j \varrho_j^{n-1} |\Gamma_j|,$$

where $\tilde{c}_j(\epsilon) = (1 + \epsilon \tilde{M}_j)^{n/2} c_j(\epsilon)^{n-1}$. Note that ϱ is Riemannian integrable since it is non-negative bounded measurable function on ∂D . Therefore, letting $\epsilon \rightarrow 0$, we obtain the desired result that

$$(6.37) \quad A(\tau) \sim \frac{\omega_{n-1} \tau^{n-1}}{(\sqrt[3]{16} \pi)^{(n-1)}} \int_{\partial D} \varrho^{n-1} ds \quad \text{as } \tau \rightarrow +\infty.$$

b) Since a metric g^{il} in C^2 can be approximated by a metric g_ϵ^{il} which is C^2 -smooth on M and piecewise $C^{2,\epsilon}$ -smooth in any compact submanifold of (\mathcal{M}, g) such that

$$|g_\epsilon^{il} - g^{il}| < \epsilon, \quad i, l = 1, \dots, n,$$

$$\left| \frac{1}{\sqrt{|g'_\epsilon|}} \frac{\partial}{\partial x_i} (\sqrt{|g'_\epsilon|} g_\epsilon^{il}) - \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} (\sqrt{|g|} g^{il}) \right| < \epsilon, \quad i, l = 1, \dots, n,$$

for all points in \bar{D} , with any given $\epsilon > 0$. In addition, any bounded domain D with C^1 -smooth boundary can also be approximated (see, the definition in Section 2) by domain D'_ϵ with C^1 -smooth and piecewise $C^{2,\epsilon}$ -smooth boundary. Thus, by Lemma 6.1 and a), we can estimate the eigenvalues for g_ϵ^{il} in D'_ϵ . But for these eigenvalues (6.37) is true, so that letting $\epsilon \rightarrow 0$ and noticing that $ds'_\epsilon \rightarrow ds$, we get that (6.37) also holds for such metric g^{il} and D . \square

Proof of Corollary 1.2. By (1.5), we have

$$(6.38) \quad A(\lambda_k) \sim \frac{\omega_{n-1} \lambda_k^{n-1}}{(\sqrt[3]{16} \pi)^{(n-1)}} (\text{vol}(\partial D)), \quad \text{as } k \rightarrow +\infty.$$

Since $A(\lambda_k) = k + 2$, we obtain (1.6), which completes the proof. \square

Proof of Theorem 1.3. Let $T_\epsilon : H^{1/2}(\partial D) \rightarrow H^{-1/2}(\partial D)$ be defined as follows: For any $\phi \in H^{1/2}(\partial D)$, we put $T_\epsilon \phi = \left(\frac{1}{(\varrho + \epsilon)^3} \frac{\partial(\Delta_g u)}{\partial \nu} \right) \Big|_{\partial D}$, where u satisfies

$$\begin{cases} \Delta_g^2 u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D, \\ u = \phi & \text{on } \partial D, \end{cases}$$

and $\epsilon > 0$ is a sufficiently small constant. Clearly, T_ϵ is a self-adjoint, elliptic, pseudo-differential operator of order 1. Also, its principal symbol has the form $c(\varrho + \epsilon)^{-3} |\xi|^3$, where $\xi \in \mathbb{R}^{n-1}$ and c is a unknown constant (we will determine it later). It is easily seen that the operator T_ϵ has the same eigenvalues $(\lambda_k(\epsilon))^3$ (respectively, the corresponding eigenfunctions u_k) as the following Steklov eigenvalue problem:

$$\begin{cases} \Delta_g^2 u_k = 0 & \text{in } D, \\ \frac{\partial u_k}{\partial \nu} = 0 & \text{on } \partial D, \\ \frac{\partial(\Delta_g u_k)}{\partial \nu} - (\lambda_k(\epsilon))^3 (\varrho + \epsilon)^3 u_k = 0 & \text{on } \partial D. \end{cases}$$

Applying the standard method (see, for example, p. 163 of [27], [11], [32] or [2]), we have

$$\begin{aligned}
A^{(\epsilon)}(\tau) &= \frac{1}{(2\pi)^{n-1}} \left(\int_{\partial D} ds \int_{c(\varrho+\epsilon)^{-3}|\xi|^3 < \tau^3} d\xi \right) + O(\tau^{(n-2)}) \\
&= \frac{1}{(2\pi)^{n-1}} \left(\int_{\partial D} ds \int_0^{\frac{(\varrho+\epsilon)\tau}{c^{1/3}}} \sigma_{n-1} r^{n-2} dr \right) + O(\tau^{(n-2)}) \\
&= \frac{\omega_{n-1}\tau^{n-1}}{(2\pi)^{n-1}} \int_{\partial D} \left(\frac{\varrho(s) + \epsilon}{c^{1/3}} \right)^{n-1} ds + O(\tau^{(n-2)}) \quad \text{as } \tau \rightarrow +\infty,
\end{aligned}$$

where $A^{(\epsilon)}(\tau) = \#\{(\lambda_k(\epsilon))^3 \leq \tau^3\}$, σ_{n-1} and $\omega_{n-1} = \frac{\sigma_{n-1}}{n-1}$ are respectively the area and volume of the unit sphere and the unit ball of \mathbb{R}^{n-1} . Letting $\epsilon \rightarrow 0$, we obtain

$$A(\tau) = \frac{\omega_{n-1}\tau^{n-1}}{(2\pi)^{n-1}} \int_{\partial D} \left(\frac{\varrho(s)}{c^{1/3}} \right)^{n-1} ds + O(\tau^{(n-2)}) \quad \text{as } \tau \rightarrow +\infty.$$

It follows from the leading asymptotic formula (1.5) of Theorem 1.1 that the constant c must be 2. Therefore,

$$A(\tau) = 2^{-(\frac{n-1}{3})} \cdot (2\pi)^{-(n-1)} \omega_{n-1} \tau^{n-1} \int_{\partial D} \varrho^{n-1}(s) ds + O(\tau^{(n-2)}) \quad \text{as } \tau \rightarrow +\infty.$$

□

Remark 6.4. It is worth noting that for the harmonic Steklov eigenvalue problem (1.2) with smooth boundary ∂D , we can also obtain a better asymptotic formula with remainder estimate than that of [26] (i.e., (1.3)). In fact, we first define the pseudodifferential operator $S_\epsilon : H^{1/2}(\partial D) \rightarrow H^{-1/2}(\partial D)$ as follows: For any $\phi \in H^{1/2}(\partial D)$, we put $S_\epsilon \phi := ((\varrho + \epsilon)^{-1} \frac{\partial v}{\partial \nu})|_{\partial D}$, where v satisfies

$$\begin{cases} \Delta_g v = 0 & \text{in } D, \\ v = \phi & \text{on } \partial D. \end{cases}$$

This is just the well-known Dirichlet-to-Neumann map, and it has the same eigenvalues as the Steklov eigenvalue problem (1.2) with ϱ being replaced by $\varrho + \epsilon$. One knows (see, for example, p. 103 of [29]) that the principal symbol of the pseudodifferential operator T is $(\varrho + \epsilon)^{-1}|\xi|$. So, we have the following asymptotic formula (see, for example, p 163 of [27], [2])

$$\begin{aligned}
B^{(\epsilon)}(\tau) &= (2\pi)^{-(n-1)} \left(\int_{\partial D} ds \int_{(\varrho+\epsilon)^{-1}|\xi| < \tau} d\xi \right) + O(\tau^{(n-2)}) \\
&= \frac{\omega_{n-1}\tau^{n-1}}{(2\pi)^{n-1}} \int_{\partial D} (\varrho(s) + \epsilon)^{n-1} ds + O(\tau^{(n-2)}) \quad \text{as } \tau \rightarrow +\infty.
\end{aligned}$$

By letting $\epsilon \rightarrow 0$, we get

$$(6.39) \quad B(\tau) = \frac{\omega_{n-1}\tau^{n-1}}{(2\pi)^{n-1}} \int_{\partial D} \varrho^{n-1}(s) ds + O(\tau^{(n-2)}) \quad \text{as } \tau \rightarrow +\infty.$$

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